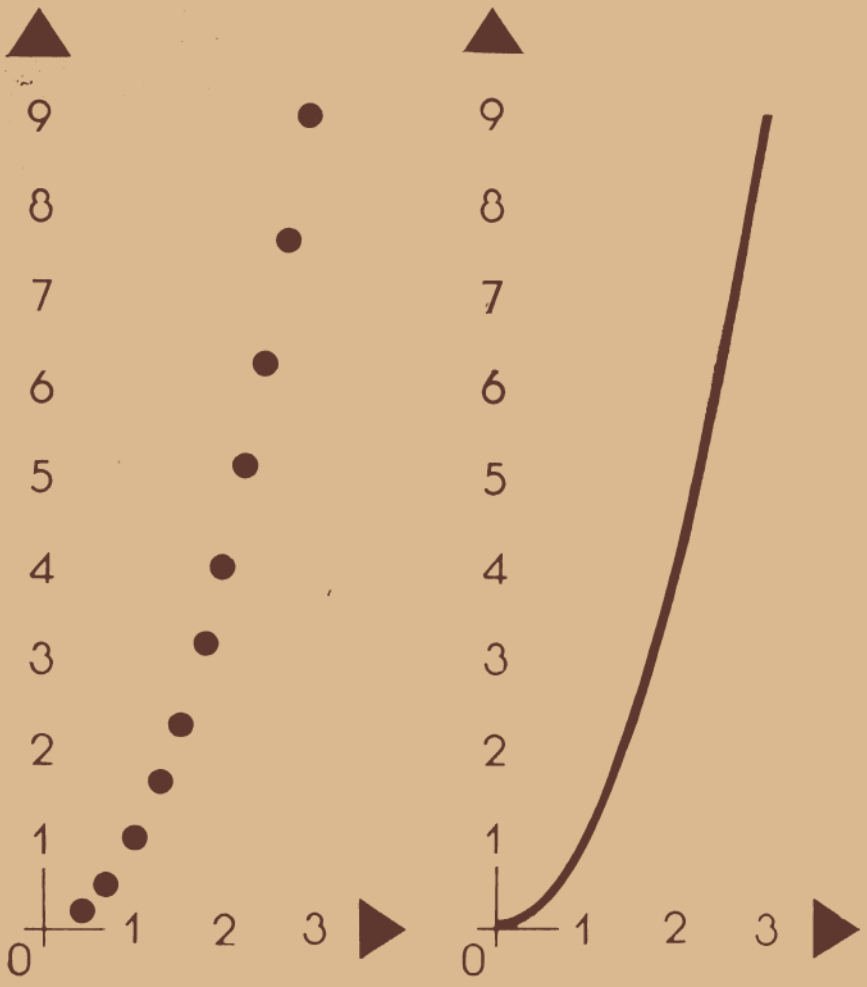
S.M.Nikolsky, M.K.Potapov

Algebra



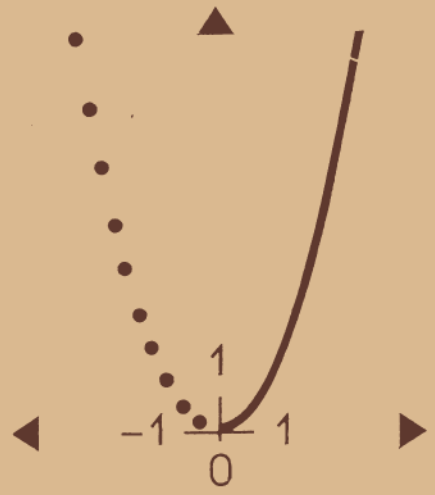
Mir Publishers Moscow

Algebra

This book is intended for self-education, but it can also be used as a study aid by pupils, pre-college students, and teachers. It reflects the mathematics taught in the 6 to 8th grades of high school. Each topic is covered in a separate section which explains the theory and each section is followed by revision questions and exercises. The answers are all given at the end of the book.

Academician Nikolsky wrote *A Course in Mathematical Analysis*, which was issued in English in 1977 by Mir Publishers and reprinted in 1981 and 1985.

Professor Potapov has written several books on mathematics, also published in English by this publishing house.



Algebra

A Study Aid for Self-education

С. М. Никольский, М. К Потапов

Алгебра

Пособие для самообразования

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PREFACE

The exposition of the development of the number concept is very difficult from the methodical point of view. When and how must the notion of real number be introduced?

Everybody agrees that a real number must be introduced as a decimal fraction, generally speaking, a nonterminating (infinite) one. But there are different points of view concerning the question: at what stage of the learning process and in what way this notion must be introduced?

We hold that the earlier the student is taught that a real number is a nonterminating decimal, the better, since even when first studying mathematics the student has to deal with the lengths of line segments, the number line, coordinate systems, graphs, and square roots. The introduction of irrationality and incommensurability with unity becomes much simpler if the student has an idea, even if it is a most elementary one, that a number is a nonterminating decimal fraction.

Our book opens with this topic. We first summarize the arithmetic which we expect is already familiar to the reader and which we will need. Adding to this knowledge, we see that a rational number is representable in the form of a repeating decimal and, conversely, that any repeating decimal is the representation of some rational number.

We should like to note that there is no necessity here to introduce the notion of a converging series. Examples of nonrepeating decimals are given and called irrational numbers.

Nonterminating decimals are compared in the same way as terminating decimals. As far as the arithmetical operations on them are concerned, we introduce approximate methods, which is important given that the syllabus requires that the student learns the elements of approximate calculation.

The notion of the polynomial, the polynomial in the standard form, in particular, and the zero polynomial is introduced in the usual way. But we stick to the algebraic point of view, that is, symbols in our book are not necessarily numbers denoted by letters. The methodical advantage of this point of view has a favourable result, especially when considering algebraic fractions. An algebraic fraction is defined as the ratio of one polynomial to another nonzero polynomial. The rules obeyed by polynomials and algebraic fractions are given. For instance, one rule covered states that an algebraic fraction can be reduced by a nonzero polynomial.

For instance, the expression $\frac{a^2-b^2}{a-b}$ is an algebraic fraction since its numerator and denominator are polynomials, $(a-b)$ being nonzero.

This fraction can be reduced by the nonzero polynomial $(a-b)$, $\frac{a^2-b^2}{a-b}=\frac{(a-b)(a+b)}{a-b}=\frac{a+b}{1}=a+b$. At this stage we also explain that if the letters in such algebraic equalities are substituted by numbers, then a true numerical equality is obtained only if both sides of the equality have sense. We return to this topic when solving equations.

The length of a line segment is determined as the number $a = \alpha_0.\ \alpha_1\alpha_2\alpha_3\alpha_4\ \ldots$, where each α represents a digit. We only formulate the statement, but we do not deem it possible to explain here why we have to consider that conversely every number $\alpha_0.\ \alpha_1\alpha_2\ldots$ is the length of a line segment.

We spend a great deal of space on graphs. Special attention is paid to linear functions and the function $y = x^2$ since we can use graphs of linear functions to solve equations and first-degree systems of equations, while the graph of $y = x^2$ helps the student understand why there are two square roots for a positive number: one positive, the other negative.

To guarantee that the square roots of a positive number y do exist, we must explain why the graph of the function $y = x^2$ is a continuous curve. We explain this in the usual way on the basis of the relationship $(x + \Delta x)^2 - x^2 = 2x\,\Delta x + \Delta x^2 \to 0\ (\Delta x \to 0)$.

Thus, we have to speak of variables tending to zero and their properties. But this material is purely descriptive and intuitive.

We also introduce the concept of the derivative and antiderivative (primitive) for a polynomial. This will turn out to be useful when studying uniformly accelerated motion during physics lessons in the eighth form.

The book contains a brief description of the fundamentals of the binary number system and also a chapter dedicated to approximate calculations.

We should like to note that much attention is paid in the text to solving linear, quadratic, and, in general, rational equations and inequalities, and also systems of these equations.

This text should prove useful for self-education, but it can also be used as a study aid for pupils, pre-college students, and as auxiliary material for teachers. Its content corresponds to the USSR syllabus of instruction for the 6-8th form of high school.

The ideas in this book were also used in a text-book we have written recently. At present this text-book is being tried out on an experimental basis at a number of schools in this country.

The Authors

Chapter 1

NATURAL NUMBERS

Sec. 1.1. Prime and Composite Numbers

The numbers

$$1;\ 2;\ 3;\ 4;\ 5;\ \ldots$$

are called *natural numbers* or *positive integers* (*whole numbers*).

We know that a sum and a product of natural numbers are also natural numbers. A difference of natural numbers is a natural number only if the subtrahend (the number subtracted) is less than the minuend (the number from which the subtrahend is subtracted). The quotient of two natural numbers is also not always a natural number.

If the quotient is a whole number, we say that the *first number is exactly divisible by the second number*. The quotient obtained by the division of one whole number by another may not be a whole number, in which case the quotient may be indicated as fraction. In this case, the *first number* (called the dividend) *is said to be not exactly divisible by the second number* (called the divisor).

For instance, 25 is exactly divisible by 5 since the quotient is a whole number, 5; 7 is not exactly divisible by 3 since there is a remainder after the division, 1. The word "exactly" is usually omitted in such sentences. Thus, we say, for instance, that 6 is divisible by 3; 30 is divisible by 5; 37 is not divisible by 4; 40 is not divisible by 6; 5 is divisible by 1.

We shall denote natural numbers by small letters p, q, ...

Each natural number p is divisible by unity

$$p : 1 = p$$

and by itself:

$$p : p = 1.$$

Among natural numbers, a special place is occupied by those which are greater than unity and are divisible only by unity and itself. Such numbers are called *prime*. Or, in other words: a prime number is a number that cannot be separated into any integral factors other than itself and 1.

Listed below are the first fifteen prime numbers:

2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47.

It is impossible to write out all prime numbers since their number is infinite. Any list of prime numbers can be extended by adding one more prime number other than the listed numbers.

Thus, every prime number is divisible only by 1 and itself.

Numbers exceeding unity which do not belong to prime numbers are called *composite*. Listed below are all composite numbers less than 30:

4; 6; 8; 9; 10; 12; 14; 15; 16; 18; 20; 21; 22; 24; 25; 26; 27; 28.

Any composite number is divisible by itself, unity (1), and at least by one more natural number. Thus, we may define composite numbers as numbers which have other divisors besides 1 and the number itself.

The number one (unity) is conventionally regarded to be neither prime nor composite. Thus, the set of natural numbers consists of prime numbers, composite numbers, and unity.

SECTION TEST

QUESTIONS

1. What is a prime number?
2. What is a composite number?
3. Does unity (number one) belong to prime numbers?
4. Does unity belong to composite numbers?

EXERCISES

1. Find out which of the following numbers are prime and which are composite:

41; 57; 1121; 793.

2. Write the first 25 prime numbers in the increasing order.
3. Write all composite numbers not exceeding 50 in the increasing order.

Sec. 1.2. Power of a Number

Sometimes, we have to multiply a number by itself several times. In this case we use a brief notation.

For instance, if the number 2 is multiplied by itself six times, then the product

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

is called the *sixth power of the number* 2 and is symbolized as 2^6, that is,

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6;$$

here, the number 6 is called the *exponent* of the power and the number 2, repeated as a factor, is called the *base*.

Analogously, the product of five factors each of which is equal to 3 is called the fifth power of the number 3 and is denoted as 3^5, that is,

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5.$$

In this example the exponent is equal to 5, the base to 3, the result being called the power.

Some more examples:

$$4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256;$$

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5 = 100{,}000.$$

Note that the first power of the number p should be understood as the number p itself, that is,

$$p^1 = p.$$

It is easy to see the validity of the following equalities:

$$2^2 \cdot 3^2 = 2 \cdot 2 \cdot 3 \cdot 3 = (2 \cdot 3) \cdot (2 \cdot 3) = 6^2;$$

$$4^3 \cdot 7^3 = 4 \cdot 4 \cdot 4 \cdot 7 \cdot 7 \cdot 7 = (4 \cdot 7)\,(4 \cdot 7)\,(4 \cdot 7) = 28^3;$$

$$5^1 \cdot 6^1 = 5 \cdot 6 = 30 = 30^1.$$

These equalities confirm the validity of the following property of powers: *the product of like powers of two numbers is equal to the same power of the product of these numbers.*

SECTION TEST

QUESTIONS

1. What is the exponent in the notation 5^9?
2. What is the product of like natural powers of three numbers equal to?

EXERCISES

1. Find the third power of the following numbers:
(a) 4; (b) 5; (c) 7; (d) 100.
2. Find the exponent of the power to which the number 2 must be raised to obtain the indicated number:
(a) 256; (b) 1024.

Sec. 1.3. Divisors of a Natural Number

The *divisor* of a given natural number is defined as a natural number by which the given number is divisible (exactly or without a remainder).

This definition implies that a prime number p has only two divisors: 1 and p, while a composite number n, in addition to the divisors 1 and n, has at least one more divisor.

Thus, the number 13 has two divisors: 1 and 13, the number 4 has three divisors: 1, 2, and 4, the number 12 has six divisors: 1, 2, 3, 4, 6, and 12.

If a divisor is a prime number, then it is called a *prime divisor*. Thus, the number 13 has one prime divisor, 13, the number 4 one prime divisor, 2, and the number 12 two prime divisors, 2 and 3.

Let us explain the following assertion: the natural number p has no other prime divisors, except 2 and 5.

Firstly, p can be equal to unity ($p = 1$). As is known, unity has no prime divisors.

Secondly, p can be a power of the number 2, that is, be equal to one of the following numbers:

$$2;\ 2^2 = 4;\ 2^3 = 8;\ 2^4 = 16;\ 2^5 = 32,\ \ldots .$$

Thirdly, p can be a power of the number 5, that is, be equal to one of the following numbers:

$$5;\ 5^2 = 25;\ 5^3 = 125;\ 5^4 = 625;\ \ldots .$$

And, finally, p can be a product of a certain power of the number 2 by some power of the number 5, for instance,

$$2^2 \cdot 5^2 = 100;\quad 2^3 \cdot 5^2 = 200;$$
$$2^4 \cdot 5 = 80;\quad 2^7 \cdot 5^4 = 80{,}000;\quad 2^3 \cdot 5 = 40.$$

SECTION TEST

QUESTIONS

1. What is a divisor of a given natural number?
2. What is a prime divisor of a given natural number?

EXERCISES

1. Find all the divisors of the following numbers:
(a) 17; (b) 45; (c) 113; (d) 476; (e) 32.
2. Find all the prime divisors of the following numbers:
(a) 19; (b) 54; (c) 112; (d) 232.
3. Write five natural numbers having no other prime divisors except 2 and 5, and five natural numbers not possessing this property.

Chapter 2

FRACTIONS

Sec. 2.1. Common and Terminating Decimal Fractions

The numbers

$$\frac{2}{3};\quad \frac{6}{8};\quad \frac{123}{150};\quad \frac{8}{4};\quad \frac{7}{5};\quad \ldots$$

may serve as examples of positive rational numbers.

In general, a *positive rational number* or a *positive common fraction* or, simply, a *fraction* is defined as a number which can be written in the form

$$\frac{p}{q},$$

where p, q are natural numbers. The number p is called the *numerator* of the fraction, and the number q its *denominator*.

Any natural number p is a particular case of fraction since it can be written in the form $\frac{p}{1}$: $p = \frac{p}{1}$.

In this chapter, we shall study positive fractions (positive rational numbers). For the sake of brevity, we shall omit the attribute "positive", still implying it. Afterwords (in Chapter 3) it will be explained how the conclusions drawn in this chapter are applied to negative numbers.

We know that the fraction $\frac{p}{q}$ will remain unchanged if both its numerator and denominator are multiplied by one and the same natural number n, in other words, for any natural number n the following equality holds true:

$$\frac{p}{q} = \frac{pn}{qn}.$$

For instance,

$$\frac{7}{3} = \frac{35}{15}; \quad 2 = \frac{10}{5}; \quad \frac{17}{2} = \frac{85}{10}; \ \ldots.$$

If the numbers p and q have no common prime divisors, then the fraction $\frac{p}{q}$ is called *irreducible*. For instance, $\frac{2}{3}$ is an irreducible fraction, while $\frac{6}{8}$ is a reducible fraction.

If $p < q$, then the fraction $\frac{p}{q}$ is called *proper*. For example, $\frac{4}{8}$ is a proper fraction, while $\frac{9}{8}$ and $\frac{8}{8}$ are not proper fractions.

If the denominator q of the fraction $\frac{p}{q}$ is equal to 10 or 100, or 1000, or 10,000, ..., that is, if q is a certain power of the number 10, then the common fraction $\frac{p}{q}$ may be written in the form of a *terminating (finite) decimal fraction*.

For instance, the common fractions

$$\frac{7}{100}; \quad \frac{12,437}{1000}; \quad \frac{234}{100,000}$$

may be written respectively in the form of the following terminating decimals:

$$0.07; \quad 12.437; \quad 0.00234.$$

Each of these terminating decimal fractions is called a *decimal expansion of the corresponding common fraction.*

In this case, we write the following equalities

$$\frac{7}{100} = 0.07; \quad \frac{12,437}{1000} = 12.437; \quad \frac{234}{100,000} = 0.00234$$

and say that the given common fractions are expanded (changed) to terminating decimals.

The reader should have in view that, for instance, $\frac{7}{100}$ and 0.07 are different notations of the same number. The first notation represents this number in the form of a common fraction, the second in the form of a terminating decimal (fraction).

It is also obvious that any terminating decimal may be written in the form of a common fraction $\frac{p}{q}$ where p is a natural number, and q is a certain power of the number 10. For instance,

$$3.0122 = \frac{30,122}{10^4}; \quad 0.00012 = \frac{12}{10^5}.$$

Thus, *if the denominator q of a common fraction* $\frac{p}{q}$ *is a certain power of the number* 10, *then this fraction may be expanded to a terminating decimal fraction.* The converse is also true: *a terminating decimal fraction represents a decimal expansion of a common fraction whose denominator is a certain power of the number* 10.

Note that a natural number may be considered as a particular case of a terminating decimal fraction. Indeed, we may write, for instance, the following equalities:

$$3 = 3.0; \quad 3 = 3.00; \quad 3 = 3.000; \ldots .$$

SECTION TEST

QUESTIONS

1. What is an irreducible fraction?
2. What is a proper fraction?
3. May a natural number be regarded as a terminating decimal?

EXERCISES

1. Find out which of the following fractions are irreducible:

$$\frac{17}{10}; \quad \frac{396}{591}.$$

2. Change the following common fractions to terminating decimals:

(a) $\frac{7,398,153}{100,000}$; (b) $\frac{1291}{100,000}$; (c) $\frac{198}{1000}$.

3. Find the common fractions equal to the following terminating decimal fractions: (a) 0.037; (b) 2.503; (c) 0.713.

Sec. 2.2. Changing a Common Fraction to a Terminating Decimal

Given below are several examples of changing a terminating decimal to an irreducible common fraction:

$$0.375 = \frac{375}{1000} = \frac{3}{8} = \frac{3}{2^3};$$

$$6.72 = \frac{672}{100} = \frac{168}{25} = \frac{168}{5^2};$$

$$0.065 = \frac{65}{1000} = \frac{13}{5^2 \cdot 2^3};$$

$$0.034 = \frac{34}{1000} = \frac{17}{5^3 \cdot 2^2};$$

$$17.0 = \frac{17}{1}.$$

From these examples it is seen that *if a terminating decimal is changed to an irreducible common fraction* $\frac{p}{q}$, *then the denominator q has no prime divisors other than* 2 *and* 5.

This assertion is also proved for the general case.

Indeed, if a terminating decimal has significant (not equal to zero) digits after the decimal point, then, as we know, it is equal to a common fraction $\frac{p}{q}$ whose denominator is a certain power of the number 10. Consequently, q has no other prime divisors than 2 and 5. Maybe we shall have to reduce this fraction, but the process of reduction will not reveal new divisors. But if a decimal has only zeros after the decimal point (as in the last example), then such a decimal fraction has the form $\frac{p}{1}$, where p is a natural number standing before the decimal point. In this case the number $q = 1$, that is, has no prime divisors (nor divisors other than 2 and 5).

But the converse is also true: *if the denominator q of an irreducible fraction* $\frac{p}{q}$ *has no other prime divisors than* 2 *and* 5, *then this fraction can be changed to a terminating decimal.*

This assertion can be readily grasped from the below examples. Indeed,

$$\frac{3}{2^2} = \frac{3 \cdot 5^2}{2^2 \cdot 5^2} = \frac{75}{10^2} = 0.75;$$

$$\frac{4}{5} = \frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10} = 0.8;$$

$$\frac{17}{5^3} = \frac{17 \cdot 2^3}{5^3 \cdot 2^3} = \frac{136}{10^3} = 0.136;$$

$$\frac{501}{2^3 \cdot 5^3} = \frac{501 \cdot 2}{2^3 \cdot 5^3} = \frac{1002}{10^3} = 1.002;$$

$$\frac{3}{2^4 \cdot 5} = \frac{3 \cdot 5^3}{2^4 \cdot 5^4} = \frac{375}{10^4} = 0.0375;$$

$$\frac{127}{1} = 127 = 127.0.$$

In the first example, to get a power of 10 in the denominator, we multiplied it by 5^2, and then the product of like powers of two numbers 2 and 5 wrote in the form of the same power of the product of these numbers, that is, of the power of the number 10. But in order for the fraction to remain unchanged, we also had to multiply the numerator by 5^2.

In the second example we multiplied the denominator by 2 and thus transformed it to the first power of the number 10. To retain the fraction unchanged, we had to multiply the numerator also by 2. We proceeded in a similar way in all the remaining examples.

The direct and converse assertions stated above can be expressed by the following words: *for an irreducible fraction* $\frac{p}{q}$ *to be changeable to a terminating decimal, it is necessary and sufficient that its denominator q have no other prime divisors than* 2 *and* 5.

There are two methods for changing an irreducible common fraction whose denominator has no other prime divisors besides 2 and 5 to a terminating decimal.

One of them was considered above. It consists in multiplying the numerator and denominator of the fraction $\frac{p}{q}$ by 2 or 5 taken in such a power as to get a certain power of the number 10 in the denominator.

The second method is the well known rule for long division, that is, for dividing numerator by denominator. Long division is the procedure of showing the subtraction and the remainder in each step of the division.

Let us, for instance, change the fraction $\frac{3}{40}$ to a decimal fraction by applying this method:

$$\begin{array}{r|l} & 0.075 \\ 40 & 3.000 \\ & 280 \\ & \overline{200} \\ & 200 \\ & \overline{0}. \end{array}$$

Hence, $\frac{3}{40}=0.075$.

Up till now we have considered decimals called terminating (or finite) decimal fractions. They are called finite because they have a finite number of digits after the decimal point. Here are some more examples of terminating decimals:

0.035; 23.2341; 7.356323; 0.00021; 35.1234567.

They have three, four, six, five, and seven digits after the decimal point, respectively.

Later on, we shall also consider nonterminating (infinite) decimals. These decimals have infinitely many digits after the decimal point.

SECTION TEST

QUESTIONS

1. What property must a common fraction possess to be changeable to a terminating decimal?
2. By what methods can a common fraction be changed to a terminating decimal fraction?

EXERCISES

1. Change the following common fractions to terminating decimals:
(a) $\frac{31}{20}$; $\frac{7}{50}$; $\frac{17}{200}$; (b) $\frac{8}{5}$; $\frac{3}{2}$; $\frac{13}{50}$, (c) $\frac{3}{150}$; $\frac{21}{35}$; $\frac{13}{65}$.
2. Find the irreducible common fraction equal to the indicated terminating decimal: (a) 0.35; (b) 2.01; (c) 7.22; (d) 0.001012.
3. Find out which of the following common fractions can be changed to terminating decimals: $\frac{1}{3}$; $\frac{24}{15}$; $\frac{12}{35}$; $\frac{5}{300}$; $\frac{1}{7}$.

Sec. 2.3. A Repeating Decimal Fraction

From the preceding section we know that any irreducible fraction $\frac{p}{q}$ is expanded into a decimal only if its denominator q has no other prime divisors than 2 and 5.

But then, if the denominator of an irreducible common fraction $\frac{p}{q}$ has a prime divisor different from 2 and 5, applying to it the rule for long division, we shall not obtain a terminating decimal.

Let us give an example. Let there be required to expand the number $\frac{7}{9}$ into a decimal fraction. This is an irreducible common fraction, and its denominator has a prime divisor 3 (different from 2 and 5). Therefore the common fraction $\frac{7}{9}$ a priori is nonexpandable into a terminating decimal fraction. Nevertheless, let us apply the known rule for long division of numerator by denominator

$$\begin{array}{r|l} & 0.777 \\ \hline 9 & 7.0000 \\ & 63 \\ \hline & 70 \\ & 63 \\ \hline & 70 \\ & 63 \\ \hline & 70 \end{array} .$$

In each step of these computations we obtain one and the same remainder 7, and the same digit 7 is added to the quotient. This process is endless (has no end). It has led us to the expression

$$0.777\ldots,$$

where the dots indicate that the digit 7 is repeated infinitely many times.

At any place to the right of the decimal point we see the same digit 7. The expression 0.777... is called a *nonterminating repeating decimal fraction* or, simply, a *repeating decimal* and is written in the following way: $0.\overline{7}$, where the digit 7 is called the *repeating block* (the *repetition*). This expression is read: "zero point seven with the 7 being the repeating block".

The number $\frac{7}{9}$ is said to have been represented in the form of a repeating decimal or to have been expanded into a repeating decimal. In this case we shall write

$$\frac{7}{9} = 0.777\ldots = 0.\overline{7}.$$

We should like to underline that the *expressions* $\frac{7}{9}$ *and* $0.\overline{7}$ *denote the same number*. That is, this number is represented in the form of either the common fraction $\frac{7}{9}$ or the nonterminating repeating decimal $0.\overline{7}$.

Let us consider another example. The common fraction $\frac{2}{99}$ is irreducible, and its denominator has prime divisors other than 2 and 5, therefore its decimal expansion cannot be finite.

Indeed,

$$\begin{array}{r|l} & 0.0202\ldots \\ \hline 99 & 2.00 \\ & 1\,98 \\ \hline & 2 \\ & \ldots \end{array}$$

This process is endless. It leads to the expression

$$0.0202\ldots,$$

where the dots now denote that the group (block) of digits $\overline{02}$ is repeated infinitely many times. This expression is also called a repeating decimal fraction and is written in the following manner: $0.\overline{02}$. This expression is read: "zero point the group of two digits $\overline{02}$ with the 02 being the repeating block".

We say that the number $\frac{2}{99}$ is represented in the form of a repeating decimal or is expanded into a repeating decimal. Here we write

$$\frac{2}{99} = 0.0202\ldots = 0.\overline{02}.$$

Using the rule for long division, we suggest to the reader to get the following equalities:

$$(1)\quad \frac{5}{9} = 0.555\ldots = 0.\overline{5};$$

$$(2)\quad \frac{17}{99} = 0.1717\ldots = 0.\overline{17};$$

$$(3)\quad \frac{101}{900} = 0.11222\ldots = 0.11\overline{2};$$

$$(4)\quad \frac{143}{45} = 3.1777\ldots = 3.1\overline{7}.$$

The right-hand sides of these equalities are read as follows:

(1) zero point five, the 5 being the repeating block;

(2) zero point seventeen, the 17 being the repeating block;

(3) zero point eleven and two, the 2 being the repeating block;

(4) three point one and seven, the 7 being the repeating block.

In all these examples we dealt with irreducible fractions whose denominators have prime divisors other than 2 and 5. The general statement is also true: if we apply the rule for long division to any

irreducible fraction $\frac{p}{q}$ whose denominator has prime divisors, other than 2 and 5, we then obtain a nonterminating repeating decimal.

Adding infinitely many zeros to a whole number or to a terminating decimal, we also turn them into nonterminating repeating decimals with the digit 0 the repeating block. For instance,

$$27 = 27.000 \ldots = 27.\overline{0}; \quad 0.354 = 0.354000 \ldots = 0.354\overline{0}.$$

Consequently, any whole number and any terminating decimal may be regarded as a particular case of a nonterminating repeating decimal.

The following statement, thus, holds true: *any positive rational number* $\frac{p}{q}$ *is expanded into a nonterminating repeating decimal.*

The converse is also true: *any repeating decimal fraction is a decimal expansion of a certain positive rational number* $\frac{p}{q}$.

Both of these statements will be justified in the next section.

SECTION TEST

EXERCISES

1. Write the decimal expansions of the following fractions:
(a) $\frac{8}{9}$; (b) $\frac{12}{99}$; (c) $\frac{17}{999}$; (d) $\frac{1}{6}$; (e) $\frac{35}{77}$.

Sec. 2.4. Repetition Property of a Decimal Expansion of a Common Fraction

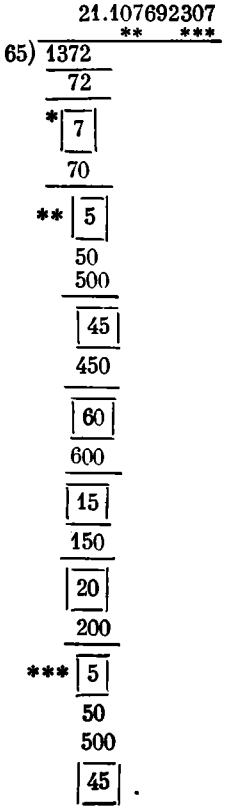
Suppose we are given an arbitrary positive rational number in the form of an irreducible fraction $\frac{p}{q}$. Let us show that if we apply to this fraction the rule for long division, we shall get, as a quotient, either its terminating or nonterminating (but repeating) decimal expansion.

We already know when a terminating decimal expansion may be obtained. For this purpose, q must not have any other prime divisors than 2 and 5. Otherwise we may obtain only a nonterminating decimal expansion. Our aim is to prove that it is repeating.

Let us first consider an example.

Suppose we have to find the decimal expansion of the common fraction $\frac{1372}{65}$. We shall divide 1372 by 65 according to the rule for long

division:



Here, the single asterisk indicates that step of work when the last digit of the dividend is brought down. The remainders obtained after this step are written in small squares. We see that the remainders marked by two and three asterisks are equal to each other. This shows that the process of division is of repeating character and leads to a nonterminating repeating decimal, that is,

$$\frac{1372}{65} = 21.1\overline{076923}.$$

The fact that the decimal expansion of our fraction must be nonterminating and repeating can be explained without computations.

The given fraction $\frac{1372}{65}$ is irreducible, and its denominator 65 has the prime divisor 13, different from 2 and 5. Consequently, the decimal expansion of the number $\frac{1372}{65}$ cannot be terminating since the remainders occurring during the process of

division are positive (not equal to zero) in any step of the work. At the same time each remainder is less than 65, that is, it equals one of the numbers: 1, 2, 3, . . ., 63, 64. But if we shall consider all the remainders in succession, beginning with the one marked by one asterisk, then among the first sixty-five of them there will necessarily occur two equal remainders which just indicates that we must obtain a repeating decimal for the quotient. This reasoning is valid for any irreducible common fraction $\frac{p}{q}$ whose denominator q has at least one prime divisor different from 2 and 5. If we divide p by q applying the rule for long division, then the step will occur when all the digits of the dividend will be brought down. If we consider in succession q remainders (all of them are positive) which occur beginning with this moment, then among them we shall always come across two equal remainders, and this is just an indication that the process of division is endless and repeating.

Thus, we have proved that *any positive rational number* $\frac{p}{q}$ *may be expanded into a nonterminating repeating decimal fraction.*

We recall here that a terminating decimal fraction is also called a repeating decimal (with the 0 being the repeating block). The converse is also true: *any repeating decimal fraction is a decimal expansion of a positive rational number.*

Below, by the way of examples we show how this number can be found.

Example 1. Write the repeating decimal

$$2.35\overline{7}$$

in the form of a common fraction.

Let us denote the desired common fraction by x, then the following equality holds true:

$$x = 2.35\overline{7}.$$

Multiplying this equality by 100 and 1000, we obtain that the following equalities are also true:

$$100x = 235.\overline{7}, \quad 1000x = 2357.\overline{7}.$$

Subtracting the second equality from the third, we find that

$$1000x - 100x = 2357 - 235,$$

whence

$$x = \frac{2357-235}{900} = \frac{2122}{900}.$$

Applying the rule for long division to the fraction $\frac{2122}{900}$ we obtain that this fraction is just equal to the repeating decimal $2.35\overline{7}$.

Example 2. Write the repeating decimal

$$0.\overline{0108}$$

in the form of a common fraction.

Let us denote the desired common fraction by x:

$$x = 0.\overline{0108}.$$

Multiplying this equality by 10,000, we get

$$10{,}000x = 108.\overline{0108}.$$

Subtracting the first equality from the second, we obtain

$$10{,}000x - x = 108 - 0,$$

whence

$$x = \frac{108-0}{9999}.$$

These examples lead to the following rule: *to write a given repeating decimal in the form of a common fraction, do the following: from the number standing to the left of the second repeating block subtract the number preceding the first repeating block and make this difference the numerator; write in denominator the digit* 9 *so many times as many digits are contained in the repeating block, and to the right of* 9's *write so many* 0's *as many digits are contained between the decimal point and the first repeating block.*

Remark. It is easily seen that

$$2.4\overline{0} = \frac{240-24}{90} = 2.4, \quad 2.3\overline{9} = \frac{239-23}{90} = 2.4.$$

Consequently

$$2.4 = 2.4\overline{0} = 2.3\overline{9}.$$

Similarly, we can show that

$$7.25 = 7.25\overline{0} = 7.24\overline{9},$$
$$0.031 = 0.031\overline{0} = 0.030\overline{9},$$
$$12.3018 = 12.3018\overline{0} = 12.301179\overline{9}.$$

Thus, repeating decimals with the 9 being the repeating block may always be replaced by corresponding terminating decimals.

Note that the decimal expansion of the common fraction $\frac{m}{n}$ obtained according to the rule for long division cannot have 9 as the repeating block. If this were so, then in the process of applying this rule after all the digits of the dividend have been brought down, there must come a moment beginning with which all the remainders would be equal to the same number k, where $k < n$, and all the digits

of the quotient would be equal to one and the same digit 9. Hence, the equality $10k = 9n + k$ would be true, that is, $9k = 9n$ and $k = n$, which is impossible, since $k < n$.

SECTION TEST

QUESTIONS

1. What common fractions are expanded into repeating decimals, the 0 being the repeating block?
2. What condition is necessary and sufficient for a common fraction to be expanded into a repeating decimal with the repeating block different from 0?

EXERCISES

1. Find the decimal expansions of the following common fractions:
(a) $\frac{4}{9}$; (b) $\frac{17}{25}$; (c) $\frac{1689}{4950}$; (d) $\frac{5}{16}$.
2. Find the repeating decimals equal to the given common fractions:
(a) $\frac{7}{40}$; (b) 19; (c) $\frac{1}{7}$; (d) $\frac{3}{11}$.
3. Write the following repeating decimals in the form of common fractions:
(a) $23.5\overline{0}$; (b) $23.5\overline{1}$; (c) $0.1\overline{13}$; (d) $0.01\overline{027}$.

Chapter 3

REAL NUMBERS

Sec. 3.1. Rational Numbers

In the preceding chapter we considered positive repeating decimals. By putting the plus sign (+) in front of these decimals, we change nothing. Therefore, for instance, we write

$$+0.\overline{4} = 0.\overline{4} = \frac{4}{9}; \quad +1.235\overline{0} = 1.235\overline{0} = 1.235 = \frac{247}{200};$$

$$+\frac{237}{99,900} = 0.00\overline{237} = +0.00\overline{237}.$$

But if we put the minus sign in front of a positive repeating decimal, then we get a negative repeating decimal. For example:

$$-0.\overline{4} = -\frac{4}{9}; \quad -1.235\overline{0} = -\frac{247}{200}; \quad -0.00\overline{237} = -\frac{237}{99,900}.$$

The left-hand sides of these equalities are called the decimal expansions of the negative common fractions written on the right.

Thus, every negative repeating decimal is a decimal expansion of a certain negative rational number, and every negative rational number has a certain negative repeating decimal fraction as its decimal expansion.

Note that the number zero can also be written in the form of a zero repeating decimal:

$$0 = 0.\overline{0} = +0.\overline{0} = -0.\overline{0}.$$

Thus, every rational number can be expanded into a repeating decimal, and every repeating decimal is a decimal expansion of a certain rational number.

We know how to expand a rational number into a repeating decimal and how to change a repeating decimal to a rational number, and in our further reasoning we shall choose for rational numbers that form of notation which will seem to be more convenient in the case under consideration.

SECTION TEST

QUESTIONS

1. What is a zero repeating decimal?
2. What is a positive repeating decimal?
3. What is a negative repeating decimal?
4. Can any rational number be expanded into a repeating decimal?

EXERCISES

1. Give five examples of negative repeating decimals.
2. Write the following rational numbers in the form of repeating decimals:
(a) $-\frac{3}{7}$; (b) $\frac{9}{16}$; (c) $-\frac{511}{90}$; (d) 0.
3. Write the following repeating decimals in the form $\frac{p}{q}$, where p is a whole number, and q a natural number:
(a) $-0.3\overline{17}$; (b) $2.\overline{0}$; (c) $-3.4\overline{5}$; (d) $-0.\overline{0}$.

Sec. 3.2. Nonterminating (Infinite) Nonrepeating Decimals. Irrational Numbers

Nonterminating nonrepeating decimals are usually considered alongside with nonterminating repeating decimals.

The expression

$$0.10110111011110...$$

may serve as an example of a nonterminating nonrepeating decimal. The digits entering this expression are arranged according to an obvious law. But no set of digits is repeated here. There is no repeating block of digits throughout this expression. This indicates that the given decimal fraction is not an expansion of a rational number.

This nonterminating decimal fraction is called a positive irrational (not rational) number. For convenience, let us denote it by the letter a:

$$a = 0.10110111011110 \ldots .$$

We shall say that the right-hand side of this equality is the decimal expansion of the number a.

In general, an arbitrary nonterminating nonrepeating decimal is called an *irrational number*.

Here are some more examples of irrational numbers:

$$b = 0.01001000100001 \ldots;$$
$$c = 2.12345678910111213 \ldots;$$
$$d = 3.012345678910111213 \ldots;$$
$$l = 1.12123123412345 \ldots .$$

Written in the number c after the decimal point are natural numbers arranged in the increasing order beginning with 1. In this way a certain law of distribution of digits in this decimal is given. In the number d the first place after the decimal point is occupied by the digit 0 followed by all the natural numbers arranged in the increasing order beginning with 1. The law of distribution of the digits in the numler l is obvious and needs no explanation.

The numbers a, b, c, d, e are examples of positive irrational numbers. Nothing is changed if the plus sign (+) is put in front of them. For instance,

$$0.010010001 \ldots = +0.010010001 \ldots,$$
$$17.123456789101112 \ldots = +17.123456789101112 \ldots .$$

But if the minus sign (—) is put in front of a positive irrational number, then a negative irrational number is obtained:

$$-a = -0.101101110 \ldots;$$
$$-b = -0.010010001 \ldots;$$
$$-c = -2.1234567891011 \ldots;$$
$$-d = -3.01234567891011 \ldots;$$
$$-l = -1.12123123412345 \ldots .$$

A number, no matter rational or irrational, positive, negative or zero, is called a *real number*.

From the aforegoing it follows that any real number may be represented in the form of a nonterminating decimal fraction.

If a decimal has no sign or, which is the same, has the plus sign, then it is positive, that is, defines a positive number. And if a decimal is preceded by the minus sign, then it is negative, that is, defines a negative number. But this rule has one exception: if in a de-

cimal fraction we have the number zero before the decimal point and all the digits after the decimal point are zeros, then irrespective of the sign of this decimal it is equal to 0.

The number to the left of the decimal point (without sign) in a nonterminating decimal will be called the *integral part of this decimal*.

In the above examples, the integral part of the number a (and the number $-a$) is equal to zero, and the integral part of the number c (and the number $-c$) is equal to 2.

The number of decimal places in a decimal fraction means the number of digits to the right of the decimal point. The first digit after the decimal point in a nonterminating decimal is called the *digit of the first decimal place*, it represents tenths, the second digit is called the *digit of the second decimal place* it represents hundredths; the third digit is called the *digit of the third decimal place*, it represents thousandths, and so on.

The real numbers α and $-\alpha$ are termed *opposite numbers*. If α is a positive number, then $-\alpha$ is negative; and if α is a negative number, then $-\alpha$ is positive; finally, if $\alpha = 0$, then $-\alpha = 0$.

The *absolute value* (or *modulus*) of a real number is defined as:
the number α itself if α is a positive number;
zero if α is zero;
the number $-\alpha$ if α is a negative number.
The absolute value of the real number α is denoted by $|\alpha|$. Thus,

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha = 0, \\ -\alpha, & \text{if } \alpha < 0. \end{cases}$$

For instance, let

$$a = 0.10110111\ldots;$$
$$b = -2.1234567891011\ldots;$$
$$c = 0.\overline{0}.$$

Then

$$|a| = 0.10110111\ldots;$$
$$|b| = -b = 2.1234567891011\ldots;$$
$$|c| = 0.$$

We should like to underline here that rational numbers have two notations. Every rational number may be written either in the form of a common fraction or in the form of its decimal expansion which is a repeating decimal fraction.

In contrast, irrational numbers have only one notation: every irrational number represents a nonterminating nonrepeating decimal, by definition.

For the time being, we have defined real numbers formally. Their definition will be complete if we give the rules for their comparison and also the rules for their addition, subtraction, multiplication, and division.

The reader knows perfectly well the rules for adding, subtracting, multiplying, and dividing terminating decimals. We may also give formal rules for addition, subtraction, multiplication, and division of nonterminating decimals, that is, of real numbers. It should be stated here that these rules for nonterminating decimals are more complicated as compared with the corresponding rules for terminating decimals since they require application of infinite processes and therefore may have only theoretical interest.

In practice, nonterminating decimals (that is, real numbers) are added, subtracted, multiplied, and divided in an approximate way. In Sec. 3.4 it will be shown how it is done.

SECTION TEST

QUESTIONS

1. What is an irrational number?
2. What is the integral part of a nonterminating decimal?
3. What is the digit of the third place in a nonterminating decimal?
4. What is the absolute value of a real number?

EXERCISES

1. Write five nonterminating nonrepeating decimals (irrational numbers).
2. Write five positive and five negative irrational numbers.
3. Find the absolute value of the indicated numbers:

(a) $-3.17\overline{2}$; (b) $+0.00\overline{1}$; (c) $+3.17\overline{2}$; (d) $-0.\overline{0}$.

Sec. 3.3. Comparing Real Numbers

Let there be given two nonterminating decimals (both of them not having 9 as the repeating block).

When comparing nonterminating decimals, one should observe the following rules.

Rule 1. Two nonterminating decimals (that is, two real numbers) are equal to each other if they have like signs, the same integral parts, and the same digits of appropriate decimal places. Otherwise, terminating decimals are regarded to be unequal.

The only exception from this rule is the number 0 which remains unchanged irrespective of the sign put in front of it:

$$0 = 0.000 \ldots = -0.000 = +0.000 \ldots .$$

Rule 2. A negative nonterminating decimal is less than zero and less than any positive terminating decimal. The number zero is less than any positive nonterminating decimal.

Rule 3. If the integral parts of two positive decimals are different, then the decimal with a greater integral part is greater. If the integral parts are equal, then it is necessary to turn our attention to the smallest decimal place for which the digits of our decimals are different. Of the two decimals, the one having the greatest digit for this decimal place is greater.

For negative decimals everything is vice versa: the one having the smaller integral part is greater, and if the integral parts are equal, then we have to compare the smallest decimal place for which the digits of the decimals under consideration are different of the two compared decimals, and the one having a smaller digit for this decimal place is greater.

If two real numbers (two nonterminating decimals) a and b are equal to each other, then we write $a = b$. If a is less than b, then we write $a < b$, or $b > a$.

Consider the following real numbers (nonterminating decimals):

$$a = 23.01001000100001 \ldots;$$
$$b = 23.\overline{03} = 23.030303 \ldots;$$
$$c = -11.232 = -11.232000 \ldots;$$
$$d = -11.23\overline{06} = -11.23060606 \ldots;$$
$$n = 0.000 \ldots;$$
$$f = 0.0000\overline{2} = 0.0000222 \ldots;$$
$$g = 23.\overline{03} = 23.030303 \ldots;$$
$$k = -0.123 = -0.123000 \ldots;$$
$$l = -11.23\overline{06} = -11.23060606 \ldots.$$

By virtue of the comparison rule, $c < d = l < k < n < f < a < b = g$.

SECTION TEST

QUESTIONS

1. When are two real numbers equal?
2. When is one real number less than the other?

EXERCISES

1. Arrange the following real numbers in a nondecreasing order?

$$a = 3.\overline{007};\quad b = 3.\overline{0008};\quad c = -2.303003000 \ldots;$$
$$d = 3.\overline{007};\quad l = -0.23\overline{1};\quad f = -0.231\overline{17}.$$

2. Write the following real numbers in a nonincreasing order:

$$a = -2.\overline{0};\quad b = 5.1\overline{01};\quad c = -0.0\overline{01};\quad d = 0.\overline{001};\quad l = +0.\overline{0}.$$

3. Show that the following inequalities are true:
(a) $0.75757 < 0.\overline{75} < 0.75758$; (b) $3.023023 < 3.0\overline{23} < 3.023024$.

Sec. 3.4. Approximate Values of Numbers

If the number a differs but little from the number a_1, then we write

$$a \cong a_1$$

and say that the number a is approximately equal to the number a_1. The symbol $\cong$ is the sign of approximate equality.

Real numbers given by nonterminating decimals are approximated by terminating decimals. The construction of a nonterminating decimal itself prompts how to choose these approximations, namely: if a nonterminating decimal a is truncated at a certain decimal place, then a terminating decimal is obtained which may be regarded to be approximately equal to the number a.

Such a decimal is called an *approximate value* of the given real number a. For example, the following terminating decimals are approximate values of the number $0.2\overline{34}$:

$$0.2;\quad 0.23;\quad 0.234;\quad 0.2343;\quad 0.23434;\quad \ldots$$

and, hence, the following approximate equalities hold true:

$$\begin{aligned}
0.2\overline{34} &\cong 0.2;\\
0.2\overline{34} &\cong 0.23;\\
0.2\overline{34} &\cong 0.234;\\
0.2\overline{34} &\cong 0.2343;\\
0.2\overline{34} &\cong 0.23434;\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

Approximate values of the number $-1.234567891011\ldots$ may be represented by the following terminating decimals:

$$-1.2;\quad -1.23;\quad -1.234;\quad -1.2345;\quad -1.23456;\quad \ldots$$

and, hence, we may write the following approximate equalities:

$$\begin{aligned}
-1.234567\ldots &\cong -1.2;\\
-1.234567\ldots &\cong -1.23;\\
-1.234567\ldots &\cong -1.234;\\
-1.234567\ldots &\cong -1.2345;\\
-1.234567\ldots &\cong -1.23456;\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Note that the more decimal places an approximate value of a given real number is taken with, the closer this approximate value to the exact number.

Now we are going to show how to approximately add, subtract, multiply, and divide real numbers. The method applied here is very

simple: the sum (difference, product, quotient) of two numbers is regarded approximately equal to the sum (difference, product, quotient) of their approximate values.

Note that the more decimal places there are in the approximate values of the two numbers we take, the closer the sum (difference, product, quotient) of these approximate values to the exact sum (difference, product, quotient) of these numbers.

Example. Let

$$a = 3.1234567\ldots; \quad b = 1.23\overline{15};$$

then we get the following approximate equalities:

$$\begin{array}{ll} a + b \cong 4.3; & a - b \cong 1.9; \\ a + b \cong 4.35; & a - b \cong 1.89; \\ a + b \cong 4.354; & a - b \cong 1.892; \\ a + b \cong 4.3549; & a - b \cong 1.8919; \\ a + b \cong 4.35496; & a - b \cong 1.89194; \\ a + b \cong 4.354971; & a - b \cong 1.891946; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}$$

In Chapter 27 we shall study in more detail the operations with approximate values of numbers and the errors of approximations connected with them.

SECTION TEST

QUESTIONS

1. What is understood by an approximate value of a given real number?
2. What is understood by an approximate value of the product of three numbers?

EXERCISES

1. Let

$$a = 2.14\overline{56}; \quad b = 0.78788788878888\ldots.$$

Find approximately $a + b$ and $a - b$ taking in a and b one digit, two digits, five digits to the right of the decimal point.

2. Let

$$a = 2.13\overline{1}, \quad b = 9.\overline{2}.$$

Find approximately ab and $\frac{a}{b}$ taking in a and b one digit, two digits to the right of the decimal point.

Sec. 3.5. Properties of Real Numbers

For any real numbers a, b, and c the following equalities hold true:

$$a + b = b + a, \tag{1}$$

$$(a + b) + c = a + (b + c), \tag{2}$$

$$ab = ba, \tag{3}$$

$$(ab)\,c = a\,(bc), \tag{4}$$

$$a\,(b + c) = ab + ac, \tag{5}$$

$$a + 0 = a, \tag{6}$$

$$a + (-a) = 0, \tag{7}$$

$$a - b = a + (-b), \tag{8}$$

$$a \cdot 1 = a, \tag{9}$$

$$a \cdot 0 = 0, \tag{10}$$

$$-a = (-1)\,a, \tag{11}$$

$$a \cdot \frac{1}{a} = 1 \text{ (if } a \neq 0), \tag{12}$$

$$a \cdot \frac{1}{b} = \frac{a}{b} \text{ (if } b \neq 0). \tag{13}$$

Equality (1) expresses the *commutative law for addition*. It states that two real numbers may be added in either order.

Equality (2) expresses the *associative law for addition*. It states that instead of adding c to the sum $(a + b)$, we may add the sum $(b + c)$ to a obtaining the same result. Note that the expression $(a + b) + c$ may also be written in such a way: $a + b + c$, that is, we may write: $(a + b) + c = a + b + c$.

Equality (3) expresses the *commutative law for multiplication*, and Equality (4) the *associative law for multiplication*. We may also write: $(ab)\,c = abc$.

Equality (5) is called the *distributive law*. This law states that if a is multiplied by the sum $(b + c)$, then we get the same result as if a is multiplied first by b and then by c and the obtained products are added.

Equality (6) states that adding zero to any number leaves the number unchanged.

Equality (7) states that the sum of opposite numbers is the number zero.

The difference $a - b$ is the number that must be added to b to get a. Equality (8) states that this number may be written in the form of the sum a and $(-b)$, where $(-b)$ is a number opposite to b.

Equality (9) states that multiplying any number by unity leaves the number unchanged.

Equality (10) states that multiplying any number by zero yields zero.

Equality (11) states that the number opposite to the number a is equal to the product of a by the number (-1).

If $a \neq 0$, then $\frac{1}{a}$ is called the *reciprocal* of a. Thus, the reciprocal of any number (except zero) is defined as the result of 1 divided by the number. Obviously, a is the reciprocal of $\frac{1}{a}$, therefore both numbers are termed *mutually reciprocal.*

Equality (12) states that the product of any number and its reciprocal is equal to 1.

Equality (13) states that if $b \neq 0$, then the quotient obtained by the division of a by b is equal to the product of a and $\frac{1}{b}$.

Remember once again that it is impossible to divide by zero, therefore the notation $\frac{a}{0}$ has no sense for any real number a, $a = 0$ included.

Sec. 3.6. Numerical Inequalities

Real numbers obey the following rules:

1. Any real numbers a and b may be related only by one of the following relationships:

$$a = b; \quad a > b, \quad a < b.$$

2. For any real numbers a and b, such that $a < b$, there is a real number c, such that $a < c$ and $c < b$, or, which is the same,

$$a < c < b.$$

3. For any real numbers a, b, and c, it follows from the relationships $a < b$ and $b < c$ that $a < c$ (the *transitive* property of inequalities).

4. For any real numbers a, b, and c, the relationship $a < b$ implies that

$$a + c < b + c.$$

In other words, the same quantity may be added to both sides of an inequality.

5. For any real numbers a and b and any positive number c, it follows from the relationship $a < b$ that

$$ac < bc.$$

In other words, both sides of an inequality may be multiplied by the same positive number.

Let us illustrate Rule 2 by a couple of examples.

If $a = 2.13\overline{4}$, $b = 2.135\overline{0}$, then $a < b$. Setting, for instance, $c = 2.1345$, we get $a < c < b$.

If $a < 0$ and $b > 0$, then $a < b$, and we may set $c = 0$ since it is obvious that in this case $a < c < b$.

If $a = -3.236$, $b = 1.17\overline{35}$, then $a < b$. Setting, for instance, $c = -2$, we get $a < c < b$.

From Rules 1 through 5 listed above we obtain the following properties of numerical inequalities:

1. If numbers a, b, c, and d are such that $a < b$ and $c < d$, then

$$a + c < b + d.$$

Indeed, from the condition $a < b$, on the basis of Rule 4, it follows that

$$a + c < b + c;$$

from the condition $c < d$, on the basis of the same rule, it follows that

$$b + c < b + d.$$

Now, applying Rule 3, we get

$$a + c < b + d.$$

2. If positive numbers a, b, c, and d satisfy the inequalities $a < b$ and $c < d$, then they satisfy the inequality $ac < bd$.

Indeed, from the condition $a < b$, taking into account that c is a positive number, by virtue of Rule 5, we get

$$ac < bc.$$

From the conditions $c < d$ and $b > 0$, on the basis of Rule 5, it follows that

$$bc < bd.$$

Now, by virtue of Rule 3, we get

$$ac < bd.$$

3. If $a < b$, then $-b < -a$.

Indeed, adding the number $(-b - a)$ to both sides of the inequality $a < b$, by Rule 4, we get

$$a + (-b - a) < b + (-b - a),$$

whence it follows that $-b < -a$.

4. If $a < b$ and c is a negative number, then $ac > bc$.

Indeed, from $a < b$ it follows that $-b < -a$; multiplying this inequality by the positive number $(-c)$, we get

$$(-b)(-c) < (-a)(-c),$$

whence

$$bc < ac.$$

5. If a and b are positive numbers and $a < b$, then

$$\frac{1}{a} > \frac{1}{b}.$$

Indeed, multiplying the inequality $a < b$ by the positive number $\frac{1}{ab}$, on the basis of Rule 5, on reducing, we get the desired inequality

$$\frac{1}{b} < \frac{1}{a}.$$

6. If a and b are positive numbers and $a < b$, then $a^2 < b^2$.

Indeed, multiplying the inequality $a < b$ by the positive number a, on the basis of Rule 5, we get

$$a^2 < ba.$$

Similarly, multiplying the inequality $a < b$ by b, we shall have

$$ab < b^2.$$

Now, by Rule 3, we get

$$a^2 < b^2.$$

In the aforegoing we used the signs of equality and strict inequality ($<$ or $>$). Sometimes, there arises the necessity to use slack inequalities.

Example. This winter the temperature in Moscow was not lower than $-30°$. If the temperature in Moscow is denoted by t, then on any day of this winter either $t > -30°$ or $t = -30°$. In such cases we customarily write $t \geqslant -30°$.

Let us define the slack inequalities $a \leqslant b$ and $a \geqslant b$.

The numerical inequality $a \leqslant b$ is regarded to be true both for $a < b$ and for $a = b$, and untrue only in the case $a > b$.

The notation $a \leqslant b$ is read as either "a is not greater than b" or "a is less than or equal to b".

The numerical inequality $a \geqslant b$ is regarded to be true both for $a > b$ and for $a = b$; it is regarded to be untrue (false) only in the case $a < b$. The notation $a \geqslant b$ is read as either "a is not less than b" or "a is greater than or equal to b".

For instance, the inequalities $5 \leqslant 6$ and $4 \leqslant 2^2$ are both true, while the inequality $9 \leqslant 7$ is untrue; the inequality $7 \geqslant 6$ is true, whereas $3 \geqslant 4$ is a false inequality.

For slack inequalities Rules 3-5 and Properties 1-6 hold true if the sign of a strict inequality is replaced by the sign of a slack inequality.

We already noted that if $a < b$ and $b < c$, then we write

$$a < b < c.$$

Similarly, if $a \leqslant b$ and $b < c$, then we write

$$a \leqslant b < c,$$

and if $a \leqslant b$ and $b \leqslant c$, then we write

$$a \leqslant b \leqslant c.$$

SECTION TEST

QUESTIONS

1. What inequalities may be added together?
2. What inequalities may be multiplied?
3. When is the inequality $a \leqslant b$ true?
4. When is the inequality $a \geqslant b$ untrue?

EXERCISES

1. For the numbers a and b find a number c such that $a < c < b$:
(a) $a = 0$, $b = 0.0123\overline{1}$; (b) $a = 2.13\overline{4}$, $b = 2.135$;
(c) $a = -1$, $b = 0.172$; (d) $a = -3.231$, $b = -1.17\overline{35}$.
2. Prove that if $a < b$ and $c < d$, then $a - d < b - c$.
3. Prove that if positive numbers a, b, c, and d satisfy the inequalities $a < b$ and $c < d$, then they also satisfy the inequality $\frac{a}{d} < \frac{b}{c}$.

Sec. 3.7. The Length of a Line Segment

Every real number may be written in the form of a nonterminating decimal fraction. If we want to write an arbitrary nonterminating decimal, this is conveniently done with the aid of letters.

Let there be given, for definiteness, a positive nonterminating decimal. Let us denote its integral part by α_0. Thus, α_0 is either 0 or some natural number. We then denote the first digit to the right of the decimal point by α_1, the second digit by α_2, the third digit by α_3, and so forth. As a result, the given decimal will have the following notation:

$$\alpha_0.\alpha_1\alpha_2\alpha_3 \ldots .$$

Putting the minus sign in front of this expression, we get the decimal $-\alpha_0.\alpha_1\alpha_2\alpha_3 \ldots .$

Fig. 1

Suppose we are given a straight line l (Fig. 1) and a point O on it. Let us take on l a unit segment OE (that is, its length is equal to unity) and an arbitrary point A.

Let q denote a natural number. We divide the unit segment into q

equal parts (small segments). Each of them is said to have the length $\frac{1}{q}$.

Let us lay off on l a segment of length $\frac{1}{q}$ from the point O towards the point A. It may happen that it goes in OA exactly p times, where p is a natural number. Then the line segment OA is said to have the length $|OA|$ equal to $\frac{p}{q}$:

$$|OA| = \frac{p}{q}.$$

Another possibility, and this will occur much more frequently, consists in that for a certain natural p the line segment of length $\frac{p}{q}$ will cover only a part of the line segment OA, while the line segment $\frac{p+1}{q}$ will contain the point A strictly within itself. Then we shall say that the number $\frac{p}{q}$ is an approximate value of the length of the line segment OA with an accuracy to $\frac{1}{q}$ with deficit and we shall be able to write only an approximate equality

$$|OA| \cong \frac{p}{q}, \tag{1}$$

adding that it is obtained with an accuracy to $\frac{1}{q}$ with deficit.

In order not to consider two cases, in the first case we shall also regard that the number $\frac{p}{q}$ is an approximate value of the length of the line segment $|OA|$ with an accuracy to $\frac{1}{q}$ with deficit.

Thus, if we are given an arbitrary natural number q, then we can determine a natural number p such that

$$\frac{p}{q} \leqslant |OA| < \frac{p+1}{q}.$$

In this sense we say that the number $\frac{p}{q}$ is an approximate value of the length of the line segment OA to within $\frac{1}{q}$ with deficit.

It is frequently assumed that q is equal to 1 or 10, or 10^2, or 10^3, ... Accordingly, the line segment is measured to within

$$1; \frac{1}{10}; \quad \frac{1}{100}; \quad \frac{1}{1000}; \quad \ldots .$$

With a rapid growth of engineering, there arises the necessity to manufacture mechanical parts, and, consequently, to measure them with a very high accuracy. For example, the accuracy of 1/1000 mil-

limetre for the workpieces machined on modern lathes is a usual thing.

Suppose we measured the line segment OA with an accuracy to 1, and it turned out that an approximate value of its length with deficit is equal to a nonnegative whole number α_0. Obviously, if we now measured the line segment OA to within 0.1 with deficit, we would obtain a decimal fraction $\alpha_0.\alpha_1$, where α_1 is some digit, and α_0 is a number which was found during the preceding step of measurement.

And if the line segment is measured with an accuracy to 0.01, then we obtain a decimal fraction

$$\alpha_0.\alpha_1\alpha_2,$$

where α_2 is a new digit, and α_0 and α_1 are already found numbers.

As accuracy becomes higher and higher, the digits already found are retained, and one more digit is added to them.

Thus, the approximate value of the length of the line segment OA to within 1 with deficit is a certain nonnegative whole number

$$\alpha_0 \cong |\, OA \,|.$$

The approximate value of the length of the line segment OA with an accuracy to 0.1 with deficit is a decimal fraction

$$\alpha_0.\alpha_1 \cong |\, OA \,|,$$

where α_0 was found above.

The approximate value of the length of the line segment OA to within 0.01 with deficit is the decimal fraction

$$\alpha_0.\alpha_1\alpha_2 \cong |\, OA \,|,$$

where the numbers α_0 and α_1 were found above, and so on.

The *length of the line segment* OA is a number

$$|\, OA \,| = a = \alpha_0.\alpha_1\alpha_2\alpha_3 \ldots, \tag{2}$$

expressed by a nonterminating decimal fraction which is defined as follows:

α_0 is a nonnegative whole number equal to the approximate value of the length of the line segment OA to within 1 with deficit;

$\alpha_0.\alpha_1$ is a decimal fraction equal to the approximate value of the length of the line segment OA to within 0.1 with deficit;

$\alpha_0.\alpha_1\alpha_2$ is a decimal fraction equal to the approximate value of the length of the line segment OA to within 0.01 with deficit, and so forth.

In practice, when measuring a line segment even with the aid of perfect methods we are able to determine only α_0 and first several digits to the right of the decimal point of the number a. But theoretically, it is possible to imagine that we can determine by measurement any number of indicated digits $\alpha_1, \alpha_2, \alpha_3, \ldots$.

Thus, an arbitrary line segment OA has the length $a = |OA|$ which is a positive number defined as was described above.

Conversely, if we are given any positive number a, then it is possible to indicate a line segment OA whose length is equal to this number:

$$|OA| = a.$$

Remark. We are not going to prove the converse statement since this involves one nonelementary axiom of geometry (the axiom on enclosed line segments).

Note that if the line segment OA is a part of the line segment O_1A_1, then

$$|OA| < |O_1A_1|.$$

SECTION TEST

QUESTIONS

1. What is an approximate value of the length of a line segment with an accuracy $\frac{1}{q}$ with deficit?
2. How is the length of a line segment defined?

Sec. 3.8. A Number Line (Scale). A Coordinate Axis

Suppose we are given a horizontal line l (Fig. 2). Let us fix on it a point O which will be called the *origin* (or the *initial point*). The point O separates l into two rays. The ray directed rightwards will be called the *positive ray*; in Fig. 2 it is marked by an arrow. The ray directed leftwards will be called the *negative ray*. In such a manner, a *positive direction* is said to be chosen on the given line.

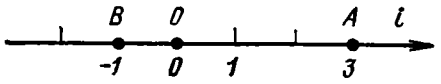


Fig. 2

Let us also take a line segment for a unit length (a unit line segment). By this we have introduced the *scale*.

A straight line with the origin and positive direction chosen and the scale introduced is called the *number line*.

Figure 2 represents a number line drawn horizontally with the positive direction extending to the right of the point O, but, generally speaking, a number line may be arranged vertically or somehow else and the positive direction on it may be chosen so as it may turn out to be convenient.

Let us associate each point of the number line with a real number x according to the following rule:

the origin O is associated with the number zero ($x = 0$);

the point A, if it lies on the positive ray, is associated with the number x equal to the length $|OA|$ of the line segment OA ($x = |OA|$);

the point A, if it lies on the negative ray, is associated with a negative number x equal to the length of the line segment OA taken with the minus sign ($x = -|OA|$).

A number line thus defined is also called the *coordinate axis x* or, simply, the ***x*-axis**.

The number x corresponding to an arbitrary point of the x-axis according to the aforegoing rule is called the *coordinate* of this point.

However, the number x in these definitions may be replaced by another letter: y, z, t, ...

Let us mention the following properties:

1. To every point of the x-axis there corresponds, in accord with the indicated rule, a real number x which is the coordinate of this point.

2. Two distinct points A and B on the x-axis have different coordinates x_1 and x_2. Here, if the point B lies to the right of the point A, then $x_1 < x_2$.

3. An arbitrary real number is the coordinate of a certain (a single one, b virtue of Property 2) point of the x-axis.

We see, that, according to a certain rule, to every point of the x-axis there corresponds one and only one real number. Moreover, by this rule, to two distinct points of the x-axis there correspond different numbers and each real number corresponds to some point of the x-axis.

These statements are replaced by the following words:

The indicated rule establishes a one-to-one correspondence between the points of the x-axis and real numbers.

The point O is called the *origin* of the coordinate axis x. The ray extending from the point O along the x-axis in the positive direction is called the *positive coordinate semiaxis x*, and the ray extending from the point O along the x-axis in the negative direction is called the *negative coordinate semiaxis x*.

For brevity, the point having the coordinate x is called the point x. For instance, in Figure 2 the point A is called point 3, and the point B point (-1).

If a point x moves along the x-axis in the positive direction and its coordinate x may attain arbitrarily large values, then this point is said to tend to plus infinity; in this case we write $x \to +\infty$.

Analogously, if a point x moves along the coordinate axis in the negative direction and its coordinate x is such that the number $|x|$ may attain arbitrarily large values, then this point is said to tend to minus infinity; in this case we write $x \to -\infty$.

SECTION TEST

QUESTIONS

1. What is a number line?
2. What is a coordinate axis?
3. What is the coordinate of a point on the coordinate axis?
4. What is meant by the notation $x \to +\infty$?
5. What is meant by the notation $x \to -\infty$?
6. How should we understand the following statement: the set of all the points of the coordinate axis is in one-to-one correspondence with the set of all real numbers?

EXERCISES

1. Mark on the number line the following numbers:

$$-\frac{7}{3}; \quad 0.\bar{3}; \quad 2.\bar{0}.$$

2. Mark on the coordinate axis x the following points:

$$x = -0.01; \quad x = 5.3; \quad x = 0.\bar{0}.$$

Sec. 3.9. Intervals

Let us be given two real numbers a and b satisfying the inequality $a < b$.

These numbers may be regarded as the coordinates of certain points of the x-axis which we agreed to term the points a and b.

A line segment with the end points a and b may be understood as the set of points consisting of its end points a and b and all the points of the x-axis lying between them. This set of points is called a *closed interval* and is symbolized by $[a, b]$. But we do not distinguish between numbers and points of the x-axis whose coordinates are represented by these numbers, therefore the notation $[a, b]$ also denotes the set of all real numbers x which satisfy the inequalities $a \leqslant x \leqslant b$.

The points a and b are the end points of the closed interval $[a, b]$, they belong to it.

If the points a and b are rejected from the closed interval $[a, b]$, then we shall get the set of points which is called an *open interval* and is denoted by the symbol (a, b). We may also say that the open interval (a, b) is the set of all real numbers x satisfying the inequalities $a < x < b$.

Example 1. The open interval $(0, 1)$ is the set of all real numbers x satisfying the inequalities $0 < x < 1$.

Example 2. The closed interval $[0, 1]$ is the set of all real numbers satisfying the inequalities $0 \leqslant x \leqslant 1$.

Above, we supposed that a and b are given numbers (or points on the x-axis). But the term "open interval" may also be understood in a broader sense by replacing b by $+\infty$ or a by $-\infty$.

The notation $(a, +\infty)$, where a is a given number means the set of all real numbers x satisfying the inequality $a < x$ or the set of points of the x-axis having abscissas $x > a$. Thus, the notation $(a, +\infty)$ means the set of points of the ray on the x-axis extending rightwards from the point a with the end point a rejected.

The notation $(-\infty, a)$, where a is a given number, means the set of all real numbers x satisfying the inequality $x < a$ or the set of points of the ray on the x-axis extending from the point a leftwards with the end point a rejected.

Finally, the notation $(-\infty, +\infty)$ denotes the set of all real numbers or the set of all the points of the x-axis.

Thus, the open interval (a, b) may be *finite* if a and b are given numbers (or points of the x-axis), but it may also be *infinite* if a and b are replaced by $-\infty$ and $+\infty$, respectively.

The closed interval $[a, b]$ is always finite since it is defined by the given numbers a and b (or the points of the x-axis).

SECTION TEST

QUESTIONS

1. What is a closed interval?
2. What is a finite open interval?
3. What is an infinite open interval?

Chapter 4

POWER WITH AN INTEGRAL EXPONENT

Sec. 4.1. Power with a Natural Exponent

In Chapter 1 (Sec. 1.2) we learned the meaning of an exponent. Now, we are going to introduce this notion in the general form.

The product of n factors each of which is equal to the number a is called the nth *power of the number* a and is denoted by a^n. Thus,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}. \qquad (1)$$

In this expression, the number a is called the *base*, the number n which indicates how many times the base is to be used as a factor is called the *exponent* of the power, and the multiplication of the number a by itself n times is termed *raising to a natural power* or *raising the number a to the* nth *power*.

In this definition, the exponent n is a natural number exceeding unity since a product has sense if it consists of at least two factors.

Therefore, it is also necessary to define the power with exponent equal to unity. By definition, the first power of the number a is equal to a:

$$a^1 = a.$$

However, we shall apply Equality (1) to the case $n = 1$ as well, that is, we shall regard the expression on the right in (1) for $n = 1$ to be equal to a.

Note that in future we shall regard raising to a natural power not as a new operation, but as a special case of multiplication.

Let us now consider the most important properties of natural powers of numbers.

Let a and b denote arbitrary real numbers, and m and n arbitrary natural numbers. Then the following equalities hold true:

$$a^m \cdot a^n = a^{m+n}, \tag{2}$$

$$(a^m)^n = a^{mn}, \tag{3}$$

$$a^n \cdot b^n = (ab)^n. \tag{4}$$

Equality (2) states that *when we multiply powers of the same number a, we add exponents of the given powers.*

For instance,

$$a^5 \cdot a^3 = a^8; \quad a^4 \cdot a = a^5; \quad a^2 \cdot a^4 = a^6; \quad a \cdot a^3 = a^4.$$

Property (2) is proved in a very simple way:

$$a^m \cdot a^n = \underbrace{(a \cdot \ldots \cdot a)}_{m \text{ times}} \underbrace{(a \cdot \ldots \cdot a)}_{n \text{ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m+n \text{ times}} = a^{m+n}. \tag{5}$$

Remark. Of course, if $m = 1$ or $n = 1$, then by $\underbrace{a \cdot \ldots \cdot a}_{m \text{ times}}$ or $\underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}$ we should understand, according to our convention, the number a.

Equality (3) states that *in raising a power (of the number a) to a power, we multiply the exponents to get the new exponent on the same base.*

For instance,

$$(a^5)^3 = a^{15}; \quad (a^3)^1 = a^3; \quad (a^1)^7 = a^7; \quad (a^2)^4 = a^8.$$

Let us prove this property first for $m = 3$, $n = 2$:

$$(a^3)^2 = (a \cdot a \cdot a)^2 = (a \cdot a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6.$$

In the general case we have to write as follows:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \ldots \cdot a^m}_{n \text{ times}} = \underbrace{\underbrace{a \cdot \ldots \cdot a}_{m \text{ times}} \cdot \underbrace{a \cdot \ldots \cdot a}_{m \text{ times}} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}}_{n \text{ times}}$$

$$= \underbrace{a \cdot a \cdot \ldots \cdot a}_{mn \text{ times}} = a^{mn}. \qquad (6)$$

Equality (4) states that the *product of the same powers of two numbers is equal to the same power of the product of these numbers*, or in other words: *the power of the product of two numbers is equal to the product of the same powers of these numbers.*

For instance,

$$2^3 \cdot 5^3 = 10^3; \quad 0.25^5 \cdot 8^5 = 2^5.$$

Let us first prove Property (4) for $n = 3$:

$$a^3b^3 = a \cdot a \cdot a \cdot b \cdot b \cdot b = (ab) \cdot (ab) \cdot (ab) = (ab)^3.$$

In more general terms the proof has the following form:

$$a^n b^n = \underbrace{(a \cdot \ldots \cdot a)}_{n \text{ times}} \cdot \underbrace{(b \cdot \ldots \cdot b)}_{n \text{ times}} = \underbrace{(ab)\,(ab) \cdot \ldots \cdot (ab)}_{n \text{ times}} = (ab)^n. \qquad (7)$$

Formulas (2), (3), (4) can be made more complicated, as is seen from the following examples.

Example 1.

$$a^m a^n a^l = (a^m a^n)\, a^l = a^{m+n} a^l = a^{m+n+l}.$$

Example 2.

$$((a^m)^n)^l = (a^{mn})^l = a^{mnl}.$$

Example 3.

$$(abc)^n = [(ab)\, c]^n = (ab)^n c^n = a^n b^n c^n.$$

Representation of a power in the form of a product of powers is often called the *factorization of a power into factors.* Several examples are given below:

$$a^3 = a \cdot a \cdot a; \quad a^5 = a^3 \cdot a^2; \quad (ab)^4 = (ab)^2 \cdot (ab)^2.$$

SECTION TEST

QUESTIONS

1. What is the nth power of the number b?
2. What is the rule for multiplying natural powers of the same number?
3. What is the rule for finding a natural power of the product of four numbers?

EXERCISES

1. Compute the following:
(a) $(0.125)^9 \cdot 8^9$; (b) $2^3 \cdot (0.7)^3 \cdot 5^3$.
2. Represent the given expression in the form of a power with base a:
(a) $(a^2)^5$; (b) $(a^4)^3$.
3. Represent a^{50} in the form of a power with the indicated base:
(a) a^2; (b) a^5; (c) a^{10}; (d) a^{25}.
4. Represent the given product in the form of a power:
(a) aba^7b^7; (b) $8 \cdot 27$; (c) $x^4y^2z^4y^2$.
5. Factor into two factors (by at least one method):
(a) 7^{10}; (b) a^6; (c) $(cd)^7$.
6. Factor into three factors (by at least one method):
(a) 5^6; (b) b^5; (c) $(mn)^4$.
7. Represent the given expression in the form of a product of powers:
(a) $(abcd)^4$; (b) $(ab^2)^3$; (c) 6^2.

Sec. 4.2. The Notion of a Power with an Integral Exponent

We know that the power of a real number a with a natural exponent n is the number a^n defined according to the rule

$$a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}.$$

However, if $n = 1$, then it is supposed that the right-hand side of this equality is equal to a:

$$a^1 = a.$$

We considered a number of properties of natural powers of numbers. In particular, it was shown that if a is a real number and m and n are natural numbers, then

$$a^m \cdot a^n = a^{m+n}.$$

Let us now consider the division of natural powers of the same number a. Here we shall regard that the number a is different from zero ($a \neq 0$) since it is impossible to divide by zero.

Let a be a nonzero real number and m and n natural numbers. Consider the quotient from the division of a^m by a^n:

$$a^m \div a^n = \frac{a^m}{a^n}.$$

Let us apply the fundamental principle of fractions to this fraction. Three cases are to be considered here: (1) $m > n$; (2) $m < n$; (3) $m = n$.

(1) If $m > n$, then

$$\frac{a^m}{a^n} = \frac{a^{m-n} \cdot a^n}{a^n} = a^{m-n},$$

and we arrive at the rule: if the number $a \neq 0$, m and n are natural numbers, and $m > n$, then

$$a^m \div a^n = a^{m-n}. \tag{1}$$

(2) Let us see what will be if $m < n$. First consider an example, let $m = 3$ and $n = 5$, then

$$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1 \cdot a^3}{a^2 \cdot a^3} = \frac{1}{a^2}.$$

We see that for $m - n$ Rule (1) is not applicable. However, if we agree to denote the fraction $\frac{1}{a^2}$ by a^{-2}, that is, to regard that

$$a^{-2} = \frac{1}{a^2},$$

then we get the equality

$$a^3 \div a^5 = a^{-2} = a^{3-5},$$

which may be considered as a particular case of Rule (1) but without limiting $m > n$.

Here is another example:

$$a^7 \div a^{10} = \frac{a^7}{a^{10}} = \frac{1}{a^3} \qquad (a \neq 0).$$

If we denote fraction $\frac{1}{a^3}$ by a^{-3}, then we get the equality

$$a^7 \div a^{10} = a^{-3} = a^{7-10},$$

which may also be considered as a particular case of Rule (1) without observing the condition $m > n$.

In the general case, if $m < n$, then

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}.$$

Here the fraction $\frac{1}{a^{n-m}}$ is denoted by $a^{-(n-m)}$.

(3) Consider now the case $m = n$. Then

$$a^m \div a^n = a^m \div a^m = \frac{a^m}{a^m} = 1.$$

This equality shows that it is expedient to regard a^0 to be equal to unity, and then we get the equality

$$a^m \div a^m = 1 = a^0 = a^{m-n},$$

which may also be regarded as a particular case of Equality (1) but without the restriction $m > n$.

The aforegoing shows that it is useful to agree to the following two conventions:

1. For any nonzero real number a and any natural number m we agree to denote the number $\frac{1}{a^m}$ by a^{-m} and to write

$$a^{-m} = \frac{1}{a^m} \qquad (a \neq 0).$$

This equality is read: "a raised to the $(-m)$ power is equal to unity divided by a raised to the mth power".

2. For any nonzero real number a, by the expression a^0 we agree to understand the number 1 and to write

$$a^0 = 1 \qquad (a \neq 0).$$

This equality is read: "a expressed with a zero exponent is equal to 1".

Thus, now we know everything about a^m, where $a \neq 0$ and m is any whole number (positive, negative, or zero). Namely, if a is any nonzero real number, then

$$a^m = \begin{cases} \underbrace{a \ldots a}_{m \text{ times}}, & \text{if } m \text{ is a natural number,} \\ 1, & \text{if } m = 0, \\ \frac{1}{a^{-m}}, & \text{if } m \text{ is a negative integer.} \end{cases}$$

Here, the number a^m is called the power with an integral exponent, the number a the base, and the number m the exponent of the power.

Remark 1. The expression 0^0 is regarded to have no sense. If m is a natural number, then the expression 0^{-m} is meaningless either but

$$0^m = \underbrace{0 \ldots 0}_{m \text{ times}} = 0.$$

Remark 2. Any positive number may be written in the form

$$a \cdot 10^k,$$

where a satisfies the inequalities $1 \leqslant a < 10$, and k is a whole number. This form is called scientific notation.

For instance,

$$273.095 = 2.73095 \cdot 10^2; \qquad 0.21 = 2.1 \cdot 10^{-1};$$
$$0.0234 = 2.34 \cdot 10^{-2}; \qquad 6781 = 6.781 \cdot 10^3.$$

In these equalities, numbers in the right-hand sides are written in scientific notation.

SECTION TEST

QUESTIONS

1. What is understood by a^{-m} if $a \neq 0$ and m is a natural number?
2. What is understood by a^0 if $a \neq 0$?
3. What is a power with an integral exponent?

EXERCISES

1. Find the quotient from the division:
(a) 5^3 by 5^2; (b) 10^5 by 10^6, (c) 7^{99} by 7^{99}.
2. Write in the form of a power with an integral exponent:
(a) $5^2 \div 5^4$; (b) $7^4 \div 7^1$; (c) $9^2 \div 3^4$.
3. Write the following numbers in scientific notation:
(a) 27.4; (b) 3821; (c) 0.0011.

Sec. 4.3. Properties of a Power with an Integral Exponent

Let a and b denote two arbitrary nonzero real numbers, and let m and n be arbitrary whole numbers. Then the following equalities hold true:

$$a^m \cdot a^n = a^{m+n}, \tag{1}$$

$$a^m \div a^n = a^{m-n}, \tag{2}$$

$$(a^m)^n = a^{mn}, \tag{3}$$

$$a^m \cdot b^m = (ab)^m, \tag{4}$$

$$a^m \div b^m = \left(\frac{a}{b}\right)^m. \tag{5}$$

Equality (1) states that *when we multiply powers of the same number a, we add exponents.*

Equality (2) states that *when we divide two quantities expressed as powers of the same base* (number a), *we subtract the exponent in the divisor from the exponent in the dividend.*

The validity of these statements is made sure easily by the way of examples. Two such examples when $m = 3$, $n = 5$ and $m = 7$, $n = 10$, were given in the preceding section. Here are some more examples ($a \neq 0$, $b \neq 0$):

(1) $a^{-3}a^2 = \frac{1}{a^3} \cdot a^2 = \frac{1}{a} = a^{-1} = a^{(-3)+2}$;

(2) $a^{-6}a^{-7} = \frac{1}{a^6} \cdot \frac{1}{a^7} = \frac{1}{a^{13}} = a^{-13} = a^{(-6)+(-7)}$;

(3) $\frac{a^{-3}}{a^{-2}} = \frac{\frac{1}{a^3}}{\frac{1}{a^2}} = \frac{a^2}{a^3} = \frac{1}{a} = a^{-1} = a^{(-3)-(-2)}$;

(4) $\frac{a^5}{a^{-2}} = \frac{a^5}{\frac{1}{a^2}} = \frac{a^5 \cdot a^2}{1} = a^7 = a^{5-(-2)}$;

(5) $\frac{a^0}{a^5} = \frac{1}{a^5} = a^{-5} = a^{0-5}$.

Equality (3) states that *in finding a power of a power* (of the number a), *we multiply the exponents to get the new exponent on the same base.*

This rule is well known to us for natural m and n. Its validity for any m and n is obvious from the following examples:

(6) $(a^{-3})^2 = \left(\frac{1}{a^3}\right)^2 = \frac{1}{a^3} \cdot \frac{1}{a^3} = \frac{1}{a^6} = a^{-6} = a^{(-3)\cdot 2}$;

(7) $(a^{-4})^0 = 1 = a^0 = a^{(-4)\,0}$;

(8) $(a^{-2})^{-3} = \left(\frac{1}{a^2}\right)^{-3} = \frac{1}{\left(\frac{1}{a^2}\right)^3} = \frac{1}{\frac{1}{a^6}} = a^6 = a^{(-2)\cdot(-3)}$.

Equality (4) states that a *product of equal powers of two numbers is equal to the same power of the product of these numbers*, or, the *power of a product of two numbers is equal to the product of the same powers of these numbers.*

Equality (5) states that a *quotient of equal powers of two numbers is equal to the same power of the quotient of these numbers*, or, the *power of a quotient of two numbers is equal to the quotient of the same powers of these numbers.*

The validity of these rules is readily understood from the following examples:

(9) $(ab)^{-3} = \frac{1}{(ab)^3} = \frac{1}{a^3 b^3} = \frac{1}{a^3} \cdot \frac{1}{b^3} = a^{-3} b^{-3}$;

(10) $\frac{a^0}{b^0} = \frac{1}{1} = 1 = \left(\frac{a}{b}\right)^0$;

(11) $\frac{a^3}{b^3} = \frac{a \cdot a \cdot a}{b \cdot b \cdot b} = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \left(\frac{a}{b}\right)^3$.

In each of the above examples (1 to 11) we gave all necessary work. But after the rules (1)-(5) are learned and practiced, the intermediate work is usually omitted referring to these rules. Thus, we write at once (assuming that $a \neq 0$, $b \neq 0$):

(1) $a^{-3} \cdot a^2 = a^{-1}$;

(2) $a^{-6} \cdot a^{-7} = a^{-13}$;

(3) $\frac{a^{-3}}{a^{-2}} = a^{-1}$;

(4) $\frac{a^5}{a^{-2}} = a^7$;

(5) $\frac{a^0}{a^5} = a^{-5}$;

(6) $(a^{-3})^2 = a^{-6}$;
(7) $(a^{-4})^0 = 1$;
(8) $(a^{-2})^{-3} = a^6$;
(9) $(ab)^{-3} = a^{-3}b^{-3}$;
(10) $\frac{a^0}{b^0} = 1$;
(11) $\frac{a^3}{b^3} = \left(\frac{a}{b}\right)^3$.

SECTION TEST

QUESTIONS

1. What is the rule for multiplying integral powers of the same number?
2. What is the rule for finding an integral power of the quotient of two numbers?
3. What is the rule for dividing integral powers of the same number?

EXERCISES

1. Represent the expression a^{50} in the form of a power with the indicated base: (a) a^{-2}; (b) a^{-5}; (c) a^{10}; (d) a^{-25}.
2. Represent in the form of a power the following products: (a) $a^{-3}b^{-3}$; (b) $7^2 \cdot 2^{-3} \cdot 7$.
3. Represent the given expression in the form of a product of powers: (a) $(a^2b^{-5})^3$; (b) $(a^{-7}b^2)^{-2}$; (c) $(a^{-3}b^{-5})^{-4}$.

Chapter 5

MONOMIALS

Sec. 5.1. Numerical Expressions

When solving many problems, we have to perform the following arithmetic operations: addition, subtraction, multiplication, and division on the given numbers. But often, before completely performing each of these operations, it is convenient to indicate beforehand the order (sequence) of performing these operations. This sequence is aimed to set up a *numerical expression* in accordance with the conditions of the problem using numbers, signs of operations, and parentheses (brackets).

Let us give several examples of numerical expressions:

$$\left(\frac{3}{2} - \frac{1}{5}\right) \cdot \frac{5}{13} - (3-5) \cdot \frac{3}{4};$$

$$(937-811) \div 63 + \frac{3-21}{9} - 2 \cdot (7 - 2^4 \div 2);$$

$$(39-15) \div 2^3 + \frac{3 \cdot 2^2}{3-7}.$$

With all the operations indicated in a numerical expression performed, we get a real number which is said to be equal to the given numerical expression.

For instance, the first numerical expression in our examples is equal to 2, the second also to 2, while the third is zero. Therefore we write

$$\left(\frac{3}{2}-\frac{1}{5}\right)\cdot\frac{5}{13}-(3-5)\cdot\frac{3}{4}=2;$$

$$(937-811)\div 63+\frac{3-21}{9}-2\cdot(7-2^4\div 2)=2;$$

$$(39-15)\div 2^3+\frac{3\cdot 2^2}{3-7}=0.$$

Note that a numerical expression indicates what arithmetic operations and in what sequence must be performed with the given numbers. Brackets are used to indicate the sequence in which the operations are to be performed (in cases when the result depends on the order of the operations). Operations indicated in brackets are to be performed first.

We, of course, suppose that all operations are possible to be performed. Let us explain these words. It is always possible to perform addition, subtraction, and multiplication of any numbers. As far as division is concerned, the reader should remember that the division of one number by another is possible only if the denominator is not equal to zero: it is impossible to divide by zero. If in a given expression a certain step requires division by zero, then this requirement is nonrealizable. Such an expression has no sense.

For instance, the expressions

$$0.37-\frac{3.1+0.172}{1.5+(2-5)\div 2};\qquad \frac{35.079}{\frac{1}{3}-0.\overline{3}}$$

have no sense since during performing the indicated operations there arises the necessity to divide by zero.

Note that a numerical expression may consist of only one number.

SECTION TEST

QUESTIONS

1. When does a numerical expression have sense?
2. When does a numerical expression have no sense?

EXERCISES

1. Which of the following expressions have sense and which have no sense? For those having sense, find the numbers they are equal to:

(a) $\dfrac{4\frac{1}{3}+5.4-0.2\overline{6}}{0.0\overline{23}-0.1} \div \left(-5+7\frac{2}{3}-2\frac{2}{3}\right)$;

(b) $3\frac{1}{7}+\left[1\frac{1}{4}\left(75 \div \frac{25}{3}-14\right)\right]\frac{4}{7}$;

(c) $\left[\dfrac{3.\overline{4}+6\frac{5}{9}}{5\frac{7}{8}-2\frac{1}{4}-0.5} \div \left(12\frac{8}{11}-8\frac{50}{99}\right)\right] \cdot \left(2\frac{3}{8}-1\frac{5}{8}\right)$.

2. Give two examples of numerical expressions indicating all arithmetic operations to be performed.

Sec. 5.2. Literal Expressions

If in a numerical expression some (or all) of its numbers are replaced by letters, then we get an example of a literal expression.

Algebra studies literal expressions, therefore they are also called *algebraic expressions.*

We are already familiar with literal expressions: an arbitrary natural number was denoted by the letter m or n, an arbitrary rational number was denoted by $\frac{p}{q}$, where p is a whole number, and q a natural number.

Let us consider other examples of literal expressions.

Example 1. From geometry it is known that the area of a rectangle is equal to the product of its base by altitude. This assertion is of the general character, it applies to any rectangle having any base and any altitude. That is why it is natural to write the corresponding equality also in the general form. This can be done using the language of literal expressions.

Let a denote the base of a rectangle, h its altitude, and S its area. Then the mentioned geometric property can be written in the following way:

$$S = ah$$

The right-hand side of this equality is a literal expression containing two letters: a and h.

The equality $S = ah$ is used as a formula by means of which we compute the area of any concrete rectangle. For instance, if a certain rectangle has the base $a = 5$ and altitude $h = 10$, then, according to this formula, its area

$$S = 5 \cdot 10.$$

The right-hand side of this equality represents a numerical expression.

Example 2. The area S of a rectangle whose altitude is longer than its base by 5 can also be written with the aid of a literal expression. Denoting by a the base of the rectangle, we get

$$S = a\,(a + 5).$$

Here a may be any positive number. The expression on the right of this equality contains only one letter a. The area of the given rectangle with the base $a = 4$ is, obviously, equal to

$$S = 4 \cdot (4 + 5).$$

Substituting the number 4 instead of a into our literal expression, we get a numerical expression.

Example 3. We know that, when divided by 2, any odd number yields the remainder 1. This enables us to write any odd number in the form

$$2n + 1,$$

where n is a whole number.

Example 4. It is obvious that any even positive number may be written in the form $2n$, where n is a natural number.

Here are some more examples of literal expressions:

$$a+(-a)-(a+3); \quad 2\alpha-\beta\gamma\cdot 0.\overline{3};$$
$$x+(y+z)^3; \quad a-(b+c)^4-d^2;$$
$$\frac{a}{b}, \quad \frac{z+a^1}{(c-d)^2}; \quad \frac{a\,(b-c)^3+1}{d+5ac}.$$

An algebraic expression may consist of only one letter, for example,

$$a; \quad c; \quad n; \quad l; \quad x; \quad z; \ldots .$$

If two given literal expressions are joined together by the sign of addition, subtraction, multiplication or division ($+$, $-$, $\cdot$, $\div$), then a new literal expression is obtained which is respectively called the sum, difference, product or quotient of the given literal expressions. However, in the case of quotient there is an exception connected with the division by zero which will be discussed later on.

For instance, the sum, difference, product, and quotient of the literal expressions

$$a + b \text{ and } a - b$$

are, in turn, literal expressions having the respective form

$$(a + b) + (a - b); \quad (a + b) - (a - b);$$
$$(a+b)\cdot(a-b) \quad \text{or} \quad (a+b)\times(a-b);$$
$$(a+b) \div (a-b) \quad \text{or} \quad \frac{a+b}{a-b}.$$

(Henceforward the multiplication sign is usually omitted.)

SECTION TEST

QUESTIONS

1. What is an algebraic expression?
2. Is it possible for an algebraic expression to consist of only one letter?

EXERCISES

1. Write an algebraic expression with the aid of which we compute the volume of a rectangular parallelepiped with edges a, b, and c.
2. Write the algebraic expression which represents:
(a) whole numbers divisible by 5;
(b) natural numbers divisible by 5 with a remainder 3.

Sec. 5.3. The Notion of a Monomial

A *monomial* may be defined as an algebraic expression indicating only one operation, that of multiplication

For instance, the literal expression

$$3aba$$

is a monomial. To obtain this monomial, we multiply together the number 3 and the letters a, b, a. Both the number 3 and these letters are said to be the *factors* of the given monomial.

Thus, a monomial may also be defined as a product of a finite number of letters and numbers, its factors.

Given below are some more monomials:

$$\alpha(-3)\beta 1\alpha 2; \quad 1a(-1)ab; \quad a0bc(-3)d1; \quad 7ada.$$

A number or a sole letter is also a monomial. For instance,

$$a; \ b; \ c; \ 1; \ -2; \ \frac{1}{3}; \ 0$$

are monomials.

The number zero is said to be a *zero monomial.*

It is supposed that monomials satisfy the properties described below (Properties 1 to 7).

Property 1. *A monomial remains unchanged if its factors are interchanged.* In other words, two monomials are regarded to be equal if they differ from each other only by the order of the factors. Thus, two monomials

$$a3bc \text{ and } 3cba$$

are equal since they differ from each other only by the order in which their factors follow.

To write the equality of two monomials, we use the sign of equality $=$. The equality thus obtained is called *algebraic.*

In what follows, we shall consider algebraic equalities, although the word "algebraic" will often be omitted, and we shall say simply "equality".

Given below are several examples of algebraic equalities:

$$a3bc = 3bca; \quad (-7)\, ccbba = (-7)\, abbcc;$$
$$2aaabb = aaa2bb = aa2bba.$$

Note that the algebraic equalities under consideration turn into true numerical equalities if the letters in them are replaced by numbers. In that case, the left-hand side of the equation will represent a product of numbers, and the right-hand side the same product with the factors interchanged; but we know that a product of numbers is independent of the order in which its factors follow.

When we say that in an equality letters are replaced by numbers, we mean that identical letters, no matter what position they occupy in the equation, are replaced by the same number.

SECTION TEST

QUESTIONS

1. What is a monomial?
2. What are the factors of a monomial?
3. What is a zero monomial?
4. How is the first property of monomials formulated?

EXERCISES

1. Write all the monomials obtained by changing the order of the factors of the indicated monomial:
(a) $3ab$; (b) $d(-2)\cdot 3c$; (c) $x7yz$.

Sec. 5.4. The Product of Monomials

The *product of monomials* is defined as a monomial whose factors are all the factors of the given monomials.

For instance, the product of the monomial $a3$ by the monomial bca is the monomial $a3bca$, which is written in the form of an algebraic equality

$$a3\cdot bca = a3bca,$$

or

$$a3 \times bca = a3bca.$$

It follows from the definition of the product of monomials that any monomial is a product of monomials each of which is one of the factors of a given monomial.

For instance, the monomial $a3bca7$ is the product of the monomials

$$a; \quad 3; \quad b; \quad c; \quad a; \quad 7,$$

that is,

$$a\cdot 3\cdot b\cdot c\cdot a\cdot 7 = a3bca7.$$

The product of k identical monomials each of which is a is briefly denoted by a^k and is called the kth power of a. The number a repeated as a factor is called the *base*; the number k which indicates how many times the base is to be used as a factor is called the *exponent* of a power. It indicates the power to which the a is to be raised.

The second power is also called the square, the third power is also called the cube (of a number).

For instance, we write

$$aa = a^2; \quad aaa = a^3; \quad aaaa = a^4; \quad \ldots$$

and respectively say that:

the product of a by a is equal to the second power of a or a squared;

the product of three factors each of which is a is equal to the third power of a or a cubed;

the product of four factors each of which is a is equal to the fourth power of a, and so on.

We also write

$$a^1 = a$$

and say that the first power of a is equal to a. Thus, the first power of a number is the number itself.

If m and n are natural numbers, then the following equality holds:

$$a^m \cdot a^n = a^{m+n}. \tag{1}$$

that is, when multiplying natural powers of the same letter, the exponents of these powers are added together.

The validity of this equality is easily seen from the following examples:

$$a^3 \cdot a^2 = aaa \cdot aa = aaaaa = a^5 = a^{3+2};$$

$$a^1 \cdot a^3 = a \cdot aaa = aaaa = a^4 = a^{1+3}$$

Equality (1) has been already known to us as pertaining to numbers, and now we have established it for letters. To simplify the notation of a monomial, identical letters are replaced by the corresponding powers of these letters. For example

$$(-3)\, aaab = (-3)\, a^3b; \quad 2aabaaa = 2a^2ba^3$$

In the last equality, the monomial on the right can be simplified still further by applying Property 1 and Equality (1):

$$2aabaaa = 2a^2ba^3 = 2a^2a^3b = 2a^5b.$$

This can be expressed as the second property of monomials.

Property 2. *In a monomial, we may replace a product of identical literal factors by the corresponding power of this letter.* In other words, two monomials are regarded to be equal if one of them is obtained

from the other by replacing a product of factors each of which is one and the same letter by the corresponding power of this letter. For instance,

$$5a^2bab^3 = 5a^3b^4; \quad 2a^3baa3b^2 = 2a^53b^3.$$

SECTION TEST

QUESTIONS

1. What is the product of monomials?
2. What is the kth power of the letter a?
3. How is the second property of monomials formulated?

EXERCISES

1. Write the following products in the form of powers:
(a) $a \cdot a \cdot a \cdot a \cdot a$; (b) $a^5a^2a^3$.
2. Using the notion of a natural power, write the monomials equal to the given one:
(a) $\alpha\beta\alpha\beta\alpha$; (b) $2 \cdot 3 \cdot a \cdot 2 \cdot a \cdot 3 \cdot a$.

Sec. 5.5. Properties of Monomials

Thus, we already know two properties of monomials. Let us consider some other properties.

Property 3. *In a monomial, we may replace some of its numerical factors by their product.* In other words, two monomials are regarded to be equal if one of them is obtained from the other by replacing some of its numerical factors by their product.

For example,

$$\begin{aligned} a \cdot 7 \cdot (-3)\, b &= a\,(-21)\, b; \\ c \cdot 2 \cdot 4b \cdot 3 \cdot 1 \cdot a &= c8b3a; \\ a \cdot (-1)\, b^2 \cdot 3 \cdot 2c &= (-6)\, ab^2c; \\ 2a^2 \cdot 3 \cdot 4bc \cdot 5 &= 2a^2 \cdot 12bc \cdot 5. \end{aligned}$$

In the third equality the product $(-1) \cdot 3 \cdot 2 = -6$ is placed in front of the monomial. By virtue of Property 1 it may occupy any place.

Property 4. *A monomial one of whose factors is zero may be replaced by the number zero.* In other words: a monomial is regarded to be equal to zero if among its factors there is a zero. We may also say: the product of a number by zero is the number zero.

For instance,

$$a\,(-1)\, b0c = 0; \quad a^20b = 0; \quad 03c^2b = 0.$$

A monomial one of whose factors is the number zero is called a *zero monomial.*

Property 5. *In a monomial, the factor* 1 *may be omitted.* Or in other words: the product of the number 1 by a monomial is equal to this monomial. For instance,

$$a^2 \cdot 1 \cdot bc = a^2bc;\ 1ab^3d = ab^3d;\ 1\alpha\beta\gamma = \alpha\beta\gamma.$$

Property 6. *If the plus sign is put in front of a monomial, then we obtain a monomial equal to the original one.* For instance,

$$+abc = abc;\ +(-7)\,a^2b = (-7)\,a^2b.$$

Property 7. *If the minus sign is put in front of a monomial, then we obtain a monomial equal to the original one multiplied by the number* (—1). *For instance,*

$$-3a^2bc = (-1)\,3a^2bc;\quad -ab = (-1)\,ab,$$
$$-(-7)\,\alpha\beta = (-1)\,(-7)\,\alpha\beta.$$

Using Property 7, we get the following equalities:

$$-(-7\alpha\beta) = -[(-1)\,7\alpha\beta] = (-1)\,(-1)\,7\alpha\beta = 1 \cdot 7\alpha\beta = 7\alpha\beta;$$
$$-(-a) = (-1)\,(-1)\,a = a$$

A given monomial and the same monomial but preceded by the minus sign are called *opposite monomials.* In order to obtain one from the other, we have to put the minus sign in front of the other or, which is the same, to multiply it by the number (—1). Thus,

$$3a^2bc \text{ and } -3a^2bc$$

are opposite monomials.

With the aid of Properties 1 to 7 we can simplify monomials.

Example 1.

$$a \cdot 2 \cdot 3 \cdot (-1) \cdot a^2 \cdot b \cdot 4 \cdot 2 \cdot b^3 = (-48)\,a^3b^4 = (-1)\,48a^3b^4 = -48a^3b^4.$$

On the basis of Property 3, we have the right to replace all the numerical factors of the given monomial by their product which is equal to the number (—48). By virtue of Property 1, the obtained numerical factor (—48) may be placed at any position. We have put it in front of the monomial.

By Property 2, the product of a and a^2 may be replaced by a^3, and the product of b and b^3 by b^4; the results thus obtained may (by virtue of Property 1) occupy any position. We have placed them in alphabetical order.

Finally, on the basis of Property 7, we obtained the last equality.

Example 2.

$$17x^2\,(-1)\,2y^23x^2y = (-102)\,x^4y^3 = -102x^4y^3.$$

Example 3.

$$17x^2 \cdot 0 \cdot 2y^3\,(-3)\,x^2y = 0$$

Example 4.

$$3 \cdot 5 \cdot (-2) \cdot 2 = -60.$$

The simplifications carried out in Examples 1-4 are called the reduction of the monomials to a standard form. A standard-form monomial is defined in the next section.

SECTION TEST

QUESTIONS

1. How are the properties of monomials formulated?
2. What monomials are called opposite?

EXERCISES

1. Simplify the following monomials:
(a) $2aba^2cb$; (b) $3ab \cdot (-3)\, ba$; (c) $5\alpha\beta \cdot 0 \cdot \gamma$;
(d) $2 \cdot 3 \cdot (-5) \cdot 2$; (e) $a \cdot a \cdot a \cdot a$.

Sec. 5.6. Standard Form of a Monomial

A nonzero monomial containing letters is said to have a *standard form* if it has only one numerical factor written in the first place and each of its letters is met in the notation only once in the form of its certain power; the letters of one and the same alphabet (Latin or Greek) being written in the order of this alphabet.

The numerical factor of a standard-form monomial is called the *coefficient* of the monomial.

For instance, the nonzero monomials

$$37a^2bc^3; \quad (-12)\, ab^2c; \quad \frac{1}{3}x^4y^2; \quad \left(-\frac{4}{7}\right) ab^2x^3y^2; \quad (-1)\,\alpha^2\beta$$

have a standard form. Their coefficients are respectively equal to the following numbers:

$$37; \quad (-12); \quad \frac{1}{3}; \quad \left(-\frac{4}{7}\right); \quad (-1).$$

If a monomial has only literal factors, that is, if a numerical factor does not enter it explicitly, then its coefficient is regarded to be equal to 1. For instance, the monomials

$$a; \ ab; \ x^2yz^2$$

are monomials of a standard form. Each of them has a coefficient equal to 1.

If the coefficient of a standard-form monomial is a negative number, then such a monomial is written in the following sequence:

the minus sign is written first, then the absolute value of the coefficient, and then the literal factors. For instance, the monomials

$$-\frac{4}{3}x^4y^2; \quad -\alpha^2\beta^3$$

are regarded to be monomials of a standard form, $\left(-\frac{4}{3}\right)$ being the coefficient of the first of them, (—1) of the second. Here we write

$$-\frac{4}{3}x^4y^2 = \left(-\frac{4}{3}\right)x^4y^2; \quad -\alpha^2\beta^3 = (-1)\alpha^2\beta^3.$$

Any real number is regarded to be a standard-form monomial. For instance, the numbers

$$-5; \quad 0; \quad \frac{7}{9}; \quad -\frac{3}{195}$$

are standard-form monomials. The following monomials are not reduced to a standard form:

$$3a^2 2bc; \quad (-1)\, ba^2d^3; \quad 7a^4ba^2b^3; \quad 0 \cdot a^2 \cdot b^2.$$

But *any monomial can be reduced to a standard form* using Properties 1-7. In other words, for any monomial there is a standard-form monomial which it is equal to.

Consider several examples of reducing a monomial to a standard form.

Example 1.

$$b^2a^2\,(-1)\,c3ab^3 4c^2 = (-12)\,a^3b^5c^3 = -12a^3b^5c^3.$$

Here, the product of all the numerical factors of the given monomial is equal to (—12). This is its coefficient. We put it in front of the letters. The product of the factors which are powers of a is equal to $a^2 \cdot a = a^3$; the product of the factors which are powers of b is equal to $b^2 \cdot b^3 = b^5$; the product of the factors which are powers of c is equal to $c \cdot c^2 = c^3$. We then arrange the letters in the alphabetical order. As a result, we get the first equality whose right-hand member represents the monomial already written in a standard form. Then, using the convention on the notation of a monomial with a negative coefficient, we write the second equality.

Example 2.

$$a^3 5b0c = 0.$$

This is a zero monomial since there is a zero among its factors. Its standard form is the number zero.

Example 3.

$$\beta\alpha\gamma = \alpha\beta\gamma.$$

Here the letters α, β, γ have been arranged in the order of the Greek alphabet.

The *degree* of nonzero monomial is defined as the sum of the exponents of all its letters.

For instance, the monomial $3a^2b$ is of third degree, $3c$ is a first-degree monomial. The degree of the monomial $a^2 2ba$ is equal to four.

By definition, a nonzero real number is regarded to be a monomial of degree zero. For instance, the monomials

$$-5; \quad 7; \quad -0.3; \quad \frac{7}{16}$$

are of degree zero.

The number zero (a zero monomial) is the only monomial whose degree is not defined.

SECTION TEST

QUESTIONS

1. What is a monomial of standard form?
2. What is the coefficient of a monomial?
3. What is the degree of a monomial?

EXERCISES

1. Reduce each of the given monomials to a standard form:

(a) $cdab$, (b) $\frac{1}{500} xy(-1)yzx^2$, (c) $\left(-\frac{4}{3}\right) xy^2 (0.3)^2 zx^4$, (d) $\beta\alpha$,

(e) $7x0y$, (f) $-\frac{7}{13}$, (g) 0.

2. Reduce the given monomial to a standard form, find its coefficient and degree:

(a) $3acb5$; (b) $dcab$; (c) $(-1)\alpha\gamma 5\beta$.

Sec. 5.7. Like Monomials

Nonzero monomials are called *like* (or *similar*) if they are equal or differ by their coefficients.

To know whether given monomials are similar (like), they must be reduced to a standard form, and then this question can be readily answered.

For instance,

$$abab^2 = a^2b^3 \text{ and } baab^2 = a^2b^3$$

are like monomials because they are equal to each other.

The following are like monomials:

$$x^2 3y 2x^2 = 6x^4y \text{ and } yx7x^3 = 7x^4y$$

since they differ only by the coefficients and are exactly alike in their letter parts.

The monomials

$$-3a^2bc \text{ and } a^2(-2)b3c = (-6)a^2bc$$

are also like monomials since they are exactly alike in their letter parts, differing only by the coefficients

There are no like monomials among the following:

$$a^2;\ b^2;\ a^3;\ 1;\ 3a^2b;\ 3ab^2$$

The *sum* of a finite number of like monomials is defined as a monomial similar to each of them whose coefficient is equal to the sum of the coefficients of the given monomials. If the sum of the coefficients is equal to zero, then the sum of the monomials is also equal to zero. For instance,

$$3a^2b + 2a^2b = (3+2)a^2b = 5a^2b;$$
$$2x^3y^2 + (-2)x^3y^2 = (2-2)x^3y^2 = 0 \cdot x^3y^2 = 0;$$
$$2abc^2 + (-abc^2) = [2+(-1)]abc^2 = 1abc^2 = abc^2;$$
$$7xyz + 3xyz + (-5)xyz = [7+3+(-5)]xyz = 5xyz.$$

The *difference* between two like monomials is defined as a monomial similar to each of them having the coefficient equal to the difference between the coefficients of the given monomials, that is, equal to the coefficient of the minuend minus the coefficient of the subtrahend.

For instance,

$$2abc^2 - 7abc^2 = (2-7)abc^2 = (-5)abc^2 = -5abc^2;$$
$$2\alpha\beta^2 - (-3\alpha\beta^2) = [2-(-3)]\alpha\beta^2 = 5\alpha\beta^2.$$

Note that the difference between two like monomials may be replaced by the sum of the minuend and the monomial opposite to the subtrahend.

For instance, if

$$7\alpha\beta - 3\alpha\beta = (7-3)\alpha\beta = 4\alpha\beta;$$
$$7\alpha\beta + (-3\alpha\beta) = [7+(-3)]\alpha\beta = 4\alpha\beta;$$

then

$$7\alpha\beta - 3\alpha\beta = 7\alpha\beta + (-3\alpha\beta).$$

Hence, indeed, the difference between two like monomials is equal to the sum of the minuend monomial and the monomial opposite to the subtrahend monomial. Therefore, henceforth, we shall speak only of the sum of like monomials.

SECTION TEST

QUESTIONS

1. What are like (similar) monomials?
2. What is the sum of like monomials equal to?

EXERCISES

1. Find like monomials among the following:
(a) a^2bc; $2abca$; a^3bc; $-3bca^2$; (b) $\alpha^3\beta$; $-\alpha\beta\alpha^2$; $-3\alpha^3\beta 0$; $7\alpha^2\beta\alpha$.
2. Find the sum of like monomials:
(a) $a^2bc + 2abca + (-3bca^2)$; (b) $(-\alpha\beta\alpha^2) + 7\alpha^2\beta\alpha + \alpha^3\beta$;
(c) $7\alpha^3 + (-3\alpha^3) + (-4\alpha^3)$.
3. Find the difference between two like monomials:
(a) $3abc - 7abc$; (b) $9a^3b^2 - 9a^3b^2$; (c) $5\alpha - 6\alpha$.

Chapter 6

POLYNOMIALS

Sec. 6.1. The Notion of a Polynomial

A *polynomial* is defined as the sum of a finite number of monomials called the *terms* of the polynomial.

The following expressions are polynomials:

$$a^2+2ab+b^2; \quad a^3+b^3; \quad \frac{1}{3}\alpha^2+(-2\beta)+(-\beta^2).$$

The first of them has the terms:

$$a^2;\ 2ab;\ b^2,$$

the second:

$$a^3;\ b^3,$$

the third:

$$\frac{1}{3}\alpha^2; \quad (-2\beta);\ (-\beta^2).$$

The third polynomial is customarily written in the following way:

$$\frac{1}{3}\alpha^2-2\beta-\beta^2 \tag{1}$$

and is also called a polynomial despite the fact that the minus sign is present in its notation. The reader should understand that calling Expression (1) a polynomial, we assume that its second term is (-2β) and its third term is $(-\beta^2)$. By virtue of this convention, the following algebraic equality takes place

$$\frac{1}{3}\alpha^2+(-2\beta)+(-\beta^2)=\frac{1}{3}\alpha^2-2\beta-\beta^2.$$

Similarly we write

$$x^3+(-y^3)=x^3-y^3;\ (-x^2)+(-y^2)=-x^2-y^2.$$

We define a monomial as an algebraic expression consisting of one term. Therefore, a monomial is also called a polynomial consisting of only one term. Due to this, the expressions

$$a^5;\quad -2ab;\ \frac{7}{3};\ -\frac{5}{9};\quad 0;\ a$$

may be regarded not only as monomials, but also as polynomials. The number zero is termed a *zero polynomial.*

Note that the above algebraic equalities turn into true numerical equalities if the letters are replaced by numbers.

For instance, if in the algebraic equality

$$x^3-y^3=x^3+(-y^3) \tag{2}$$

we set

$$x=2,\quad y=3,$$

then we get a true numerical equality

$$8-27=8+(-27).$$

But we could substitute into the given algebraic equality (2) any numbers instead of x and y, and, obviously, obtain true numerical equalities.

We should like to note once again that when we say that in an equality letters are replaced by numbers, we mean that identical letters, wherever they are found in the equality, are substituted by one and the same number.

SECTION TEST

QUESTIONS

1. What is called a polynomial?
2. What are the terms of a polynomial?

EXERCISES

1. Write out all the terms of the following polynomials:
(a) $2x^2-3xy-x^3y+7y$; (b) $-x^7-x^5-2x^3-3x$.

Sec. 6.2. Properties of Polynomials

We shall transform polynomials according to definite rules which will be called the properties of polynomials.

Property 1. *The terms of a polynomial may be interchanged.*

In other words, two polynomials are regarded to be equal if they differ from each other only by the order of their terms.

For instance, we may write the following algebraic equalities:

$$2a^2b + 3ab^2 = 3ab^2 + 2a^2b;$$
$$a^2 - b^2 = -b^2 + a^2;$$
$$x^2 - x + 1 = 1 - x + x^2 = -x + x^2 + 1.$$

Property 2. *If a zero (or a zero polynomial) is added to a polynomial, the latter remains unchanged.*

In other words, two polynomials are regarded to be equal if one of them is obtained from the other by adding the number zero.

For instance,

$$a^4 + (-a^2) + 0 = a^4 + (-a^2);$$
$$0 + \alpha\beta\gamma = \alpha\beta\gamma;$$
$$2\alpha - 3\beta + 0 - \gamma = 2\alpha - 3\beta - \gamma.$$

Property 3. *In a polynomial, we may collect (or combine) like (or similar) terms.*

In other words, two polynomials are regarded to be equal if one of them is obtained from the other by replacing like terms by their sum.

For instance,

$$a^2 + ab + ab + b^2 = a^2 + 2ab + b^2;$$
$$a^2 + ab - ab + b^2 = a^2 + b^2 + 0 = a^2 + b^2;$$
$$a^3 - 2a^2b + 2ab^2 - a^2b + ab^2 - b^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

We should like to note that like terms are collected here according to the rule formulated in Sec. 5.7, namely:

$$ab + ab = 1 \cdot ab + 1 \cdot ab = (1 + 1)\, ab = 2ab;$$
$$ab - ab = 1 \cdot ab + (-1) \cdot ab = (1 - 1)\, ab = 0 \cdot ab = 0;$$
$$-2ab^2 - a^2b = (-2) \cdot a^2b + (-1) \cdot a^2b$$
$$= (-2 - 1)\, a^2b = (-3)\, a^2b = -3a^2b;$$
$$2ab^2 + ab^2 = 2ab^2 + 1ab^2 = (2 + 1)\, ab^2 = 3ab^2.$$

Using these properties, we can simplify polynomials. The examples considered in the present section are just examples of simplifying polynomials. Here are some more examples:

$$a^3 - 4a^3 + a^4 - ba^2 + 3ba^2$$
$$= a^4 + a^3 - 4a^3 + 3ba^2 - ba^2 = a^4 - 3a^3 + 2ba^2;$$
$$x - y + x + 3y = 2x + 2y;$$
$$x^2 + 3xy - 2xy + y^2 = x^2 + xy + y^2.$$

SECTION TEST

QUESTIONS

1. What are the rules for transforming polynomials?

EXERCISES

1. Simplify the following polynomials:
(a) $xyx - 2x^2y + 2x - 3x$;
(b) $\beta\alpha^2 - 3\alpha^3 + 7\alpha\beta\alpha + 3\alpha^3 - 8\alpha^2\beta$.

Sec. 6.3. Polynomial of Standard Form

A polynomial is said to have a *standard form* if all of its terms are written in a standard form and there are no like terms among them.

Given below are examples of polynomials of a standard form:

$$2; \quad a; \quad a + b; \quad a - b;$$
$$a^2 + 2ab + b^2; \quad \frac{1}{7} - a; \quad x - y + 1.$$

The following expressions may serve as examples of nonstandard polynomials:

$$a \cdot a - 5a + 6; \quad \alpha^3 - 2\alpha\beta + \beta^2 - 3\alpha\beta - 11; \quad 3 - 5 + a^2,$$

since in the first of them not all of the terms are written in a standard form, and the second and third polynomials have like terms (the second and fourth in the second polynomial, the first and second in the third).

A polynomial of a standard form consisting of two terms is usually called a *binomial*; a polynomial of a standard form consisting of three terms is often called a *trinomial*, and so forth.

Here are relevant examples:

binomials: $\frac{1}{3}\alpha^2 - 2\beta$; $ab - cd$;

trinomials: $3a - 2b - 7$; $x + yz - 2z^2$;

quadrinomials: $\alpha + \beta - \gamma - \delta$; $-abc - acd - bcd - abd$.

Remember that a monomial is also called a polynomial consisting of only one term.

Any polynomial can be reduced to a standard form.

In other words, for any polynomial there is a polynomial of a standard form which it is equal to. To this end, it suffices to reduce each of its terms to a standard form and then to collect like terms.

For example,

$$-a\cdot a+2a(-b)+3(-a)(-2b)-2ba-71+a^2-a^2$$
$$=-a^2-2ab+6ab-2ab-71+a^2-a^2=-a^2+2ab-71;$$
$$a^3+2aba+b^2a+ba^2-2abb-b^2b$$
$$=a^3+2a^2b+ab^2+a^2b-2ab^2-b^3=a^3+3a^2b-ab^2=b^3.$$

In the second example, we reduced all the terms of the given polynomial to a standard form (the first equality). Then, in the obtained polynomial we underlined by one and two lines two pairs of like terms. On collecting like terms (the second equality), we got a polynomial of a standard form.

Remark. If a polynomial, on reducing it to a standard form, becomes zero, then it will also be called a *zero polynomial.*

For instance,

$$a-a;\quad 3x^2-x^2-2x^2$$

are zero polynomials, since, when reduced to a standard form, they turn out to be equal to zero:

$$a-a=0;\quad 3x^2-x^2-2x^2=0.$$

The highest degree of the monomials entering a polynomial when it is reduced to a standard form is called the *degree of the polynomial.* For instance, the polynomial

$$\frac{1}{3}\alpha^2-2\beta+7$$

has degree two, since it is written in a standard form and the monomials entering this polynomial have degrees 2, 1, and 0, respectively; the polynomial

$$2x-5$$

has degree 1; sometimes this polynomial is said to be a polynomial of degree 1 with respect to the letter x.

Similarly, the polynomial

$$2a-3b+7$$

is a polynomial of degree 1 with respect to the letters a and b. Any real number other than zero is a polynomial of degree zero. The number zero is the only polynomial whose degree is not defined.

SECTION TEST

QUESTIONS

1. What is a polynomial of a standard form?
2. What polynomials can be reduced to a standard form?
3. What is the degree of a nonzero polynomial?

EXERCISES

1. Find out which of the following polynomials have a standard form:
(a) $x^5 - 2x^4 + 3x^3 - 1$; (b) $\beta\alpha - \alpha^2\beta - \alpha^3\beta$; (c) $a^3b + ab^3 - a^2b^2 + 2ba b^2$.
2. Reduce the following polynomials to a standard form and determine their degree:
(a) $7a^3b - 8abc^2 + 3a^2 - 4b$; (b) $x^5 - 7y^2 + 3xyx^4 + 2x - 1$;
(c) $\alpha\gamma + 2\alpha\beta\gamma - 7\alpha^2 + 3\gamma\alpha - 3\gamma\alpha\beta$.

Sec. 6.4. The Sum and Difference of Polynomials

The *sum of polynomials* is a polynomial whose terms include all the terms of the polynomials added together.

For instance, the sum of the polynomials

$$a^2 + ab \text{ and } b^2 + ac$$

is the polynomial

$$a^2 + ab + b^2 + ac,$$

which may be written in the form of the algebraic equality,

$$(a^2 + ab) + (b^2 + ac) = a^2 + ab + b^2 + ac.$$

Here are some more examples of adding polynomials:

$$(x - y) + (y - z) + (z - u) = x - y + y - z + z - u = x - u;$$
$$(-x^2 + y^2) + (x^2 - y^2) = -x^2 + y^2 + x^2 - y^2 = 0.$$

In these equalities, the polynomials obtained as the result of the addition are reduced to a standard form.

The *difference between two polynomials* is a polynomial whose terms include all the terms of the minuend and all the terms of the subtrahend taken with reversed signs.

For instance, the difference between the polynomials

$$a^2 + ab \text{ and } b^2 + ac$$

is the polynomial

$$a^2 + ab - b^2 - ac,$$

that is, we may write:

$$(a^2 + ab) - (b^2 + ac) = a^2 + ab - b^2 - ac.$$

Here are some more examples of subtracting polynomials:

$$(\alpha^2 - \alpha\beta) - (\alpha\beta - \beta^2) = \alpha^2 - \alpha\beta - \alpha\beta + \beta^2 = \alpha^2 - 2\alpha\beta + \beta^2;$$
$$(a + b) - (c - d + a) = a + b - c + d - a = b - c + d.$$

In these equalities, the polynomials obtained as the result of the subtraction are reduced to a standard form.

Of the algebraic equalities

$$(a^2 + ab) + (b^2 + ac) = a^2 + ab + b^2 + ac;$$
$$(a^2 + ab) - (b^2 + ac) = a^2 + ab - b^2 - ac$$

we say that, when passing from their left-hand sides to the right-hand sides, we omit the parentheses.

The following rule for omitting parentheses should be strictly observed:

To omit parentheses preceded by a plus sign (+), *remove this plus sign and the parentheses without changing the sign of any term within the parentheses*; *to omit parentheses preceded by a minus sign* (—), *remove this minus sign and change the sign of each term within the parentheses.*

For instance, using this rule, we get the following algebraic equality:

$$(a - b) + (d - c) - (\alpha - \beta) - (-\gamma - \delta) = a - b + d - c - \alpha + \beta + \gamma + \delta.$$

Note that if there is no sign in front of parentheses, then the plus sign is meant. Speaking of the above equalities, we say that, when passing from their right-hand sides to the left-hand sides, we carry out the enclosing in parentheses. In doing so, observe the following rule:

To enclose a polynomial in parentheses preceded by a plus sign (+), *write in the parentheses all of its terms*; *to enclose a polynomial in parentheses preceded by a minus sign* (—), *write in the parentheses all of its terms with reverse signs.*

For instance,

$$\alpha - \beta - \gamma + d = (\alpha - \beta) + (-\gamma + d);$$
$$\alpha - \beta - \gamma + d = (\alpha - \beta) - (\gamma - d).$$

SECTION TEST

QUESTIONS

1. What is the sum of polynomials?
2. What is the difference between two polynomials?
3. What is the rule for omitting parentheses?
4. What is the rule for enclosing a polynomial in a set of parentheses?

EXERCISES

1. Transform the following sums and differences of polynomials into a polynomial of a standard form:
   (a) $(x^2 + 4x) + (x^2 - x + 1) - (x^2 - x)$;
   (b) $(a^5 + 5a^2 + 3a - a) - (a^3 - 3a^2 + a)$;
   (c) $(x^2 - 3x + 2) - (-2x - 3)$;
   (d) $(abc + 1) + (-1 - abc)$.

2. Enclose the first two terms of the given polynomials in parentheses preceded by a minus sign and the last two in parentheses preceded by a plus sign:
(a) $x^2 - y^2 + 2x - 1$; (b) $9y^2 - 1 - x^2 - 6y$; (c) $-a^3 - 3a^2 + 4 - a$.

Sec. 6.5. The Product of a Monomial by a Polynomial

The *product of a monomial by a polynomial* is a polynomial whose terms are all products of this monomial by the terms of the given polynomial.

For instance, the product of the monomial

$$\alpha\beta$$

by the polynomial

$$\alpha^4\beta - \alpha^3 + \beta^2$$

is the polynomial

$$\alpha\beta\alpha^4\beta - \alpha\beta\alpha^3 + \alpha\beta\beta^2.$$

This is written in the form of algebraic equality:

$$\alpha\beta\,(\alpha^4\beta - \alpha^3 + \beta^2) = \alpha\beta\alpha^4\beta - \alpha\beta\alpha^3 + \alpha\beta\beta^2 = \alpha^5\beta^2 - \alpha^4\beta + \alpha\beta^3. \quad (1)$$

In the last equality the polynomial is reduced to a standard form.

Here is another example of multiplication of a monomial by a polynomial:

$$-3ab\,(a^2 - 2ab + b^2) = (-3ab)\,a^2 + (-3ab)\,(-2ab) + (-3ab)\,b^2 = -3a^3b + 6a^2b^2 - 3ab^3. \quad (2)$$

When written in reverse order, Equality (1) has the form

$$\alpha^5\beta^2 - \alpha^4\beta + \alpha\beta^3 = \alpha\beta\,(\alpha^4\beta - \alpha^3 + \beta^2). \quad (3)$$

In this case the polynomial $\alpha^5\beta^2 - \alpha^4\beta + \alpha\beta^3$ is transformed into the product of the monomial $\alpha\beta$ and polynomial $\alpha^4\beta - \alpha^3 + \beta^2$.

To transform a polynomial into the product of a monomial by a polynomial means to take out a common factor of the polynomial.

In Equality (3) we took out the common factor (monomial) $\alpha\beta$ of the polynomial $\alpha^5\beta^2 - \alpha^4\beta + \alpha\beta^3$. If read from right to left, Equality (2) represents the taking out of the common factor $(-3ab)$ of the polynomial $-3a^3b + 6a^2b^2 - 3ab^3$. But we could factor out the common factor $3ab$ as well, and then we would obtain

$$-3a^3b + 6a^2b^2 - 3ab^3 = 3ab\,(-a^2 + 2ab - b^2).$$

Here is one more example of taking out a common factor:

$$x^4y - x^2y^3 = x^2y\,(x^2 - y^2).$$

Note that, multiplying the polynomial by the number (—1), we obtain a polynomial whose terms are the terms of the given polynomial with signs reversed; this new polynomial is said to be *opposite* to the given polynomial.

For instance,

$$\alpha\beta - \beta^3 \text{ and } -\alpha\beta + \beta^3$$

are opposite polynomials.

It is easily seen that the sum of opposite polynomials is equal to zero. For example,

$$(\alpha\beta - \beta^3) + (-\alpha\beta + \beta^3) = \alpha\beta - \beta^3 - \alpha\beta + \beta^3 = 0.$$

It is also easily seen that the difference between two polynomials is the sum of the minuend and the polynomial opposite to the subtrahend. For example,

$$(a + b) - (a - b) = (a + b) + (-a + b). \tag{4}$$

Indeed, by the rule for subtracting polynomials,

$$(a + b) - (a - b) = a + b - a + b = 2b,$$

by the rule for adding polynomials, we get

$$(a + b) + (-a + b) = a + b - a + b = 2b,$$

whence we obtain Equality (4).

Note also that the difference between two polynomials possesses the property that if the subtrahend is added to it, then the minuend is obtained as the result. For instance,

$$[(a + b) - (a - b)] + (a - b) = (a + b - a + b) + (a - b)$$
$$= a + b - a + b + a - b = a + b.$$

Finally, note that a polynomial multiplied by the number 1 remains unchanged. For instance,

$$1 \cdot (a^2 + b^3) = 1 \cdot a^2 + 1 \cdot b^3 = a^2 + b^3.$$

SECTION TEST

QUESTIONS

1. What is the rule for multiplying a monomial by a polynomial?
2. What is called the taking out of a common factor of a polynomial?
3. What polynomials are called opposite?

EXERCISES

1. Multiply a monomial by a polynomial:
(a) $(-abc)\ (ab + ac + bc)$; (b) $ac\ (a + 12c)$; (c) $(-2b)\ (a^2 + 2ab + 4b^2)$.
2. Take out a common factor in each of the given polynomials:
(a) $a^3 + 4b^2a$; (b) $-2a^2b + 4ab^2 - 4b^3$; (c) $-5x^3y^2 - 5x^2y$.

Sec. 6.6. The Product of Two Polynomials

The *product of two polynomials* is a polynomial whose terms are all products of each term of one of these polynomials by each term of the other polynomial.

Using this rule, we find the product of two polynomials $a^2 - ab$ and $b^2 - ab$:

$$\begin{aligned}(a^2 - ab)(b^2 - ab) &= a^2b^2 + (-ab)\,b^2 + a^2(-ab) + (-ab)(-ab)\\ &= a^2b^2 - ab^3 - a^3b + a^2b^2 = -a^3b + 2a^2b^2 - ab^3. \quad (1)\end{aligned}$$

In the second and third equalities the obtained polynomial is reduced to a standard form.

Here are some more examples of the products of polynomials:

$$\begin{aligned}(a + b)(a - b) &= aa + ba + a(-b) + b(-b)\\ &= a^2 + ab - ab - b^2 = a^2 - b^2;\\ (a^2 - xy)(y^2 - a^2x)&\\ &= a^2y^2 + (-xy)\,y^2 + a^2(-a^2x) + (-xy)(-a^2x)\\ &= a^2y^2 - xy^3 - a^4x + a^2x^2y.\end{aligned}$$

And if we have to find the product of several polynomials, then using the above rule, first find the product of any two of them, then find the product of the obtained polynomial by a third polynomial and so on.

For example,

$$\begin{aligned}(\alpha - \beta)(2\alpha + \beta)(3\alpha - 2\beta)&\\ &= (\alpha 2\alpha - \beta 2\alpha + \alpha\beta - \beta\beta)(3\alpha - 2\beta)\\ &= (2\alpha^2 - \alpha\beta - \beta^2)(3\alpha - 2\beta)\\ &= 2\alpha^2 3\alpha - \alpha\beta 3\alpha - \beta^2 3\alpha - 2\alpha^2 2\beta + \alpha\beta 2\beta + \beta^2 2\beta\\ &= 6\alpha^3 - 3\alpha^2\beta - 3\alpha\beta^2 - 4\alpha^2\beta + 2\alpha\beta^2 + 2\beta^3\\ &= 6\alpha^3 - 7\alpha^2\beta - \alpha\beta^2 + 2\beta^3. \quad (2)\end{aligned}$$

When written in reverse order, Equalities (1) and (2) have the form

$$-a^3b + 2a^2b^2 - ab^3 = (a^2 - ab)(b^2 - ab), \quad (3)$$

$$6\alpha^3 - 7\alpha^2\beta - \alpha\beta^2 + 2\beta^3 = (\alpha - \beta)(2\alpha + \beta)(3\alpha - 2\beta) \quad (4)$$

and can serve as examples of factorization of polynomials into factors.

The *factorization of a polynomial into factors* is defined as its transformation into the product of two or several polynomials.

Any polynomial can be factored into two factors one of which is a nonzero number. For instance,

$$x^2 + y^2 = 3\left(\frac{1}{3}x^2 + \frac{1}{3}y^2\right);$$

$$3a^2 - 2ab + b^2 = 5\left(\frac{3}{5}a^2 - \frac{2}{5}ab + \frac{1}{5}b^2\right).$$

But these are trivial (obvious) factorizations into factors one of which is of degree zero, the other of the same degree as the original polynomial. Of higher interest are nontrivial factorizations, that is, those in which each of the obtained factors has a degree exceeding zero. Such factorizations are exemplified by (3) and (4).

Remark 1. If it is necessary to multiply together polynomials of nonstandard form, then it is natural to reduce them to a standard form and then apply the rule for multiplication of polynomials. The result will be the same, but the work, as a rule, will be considerably reduced.

For example,

$$(a^2 - ab + ab - b^2)(2a - b - a) = (a^2 - b^2)(a - b)$$
$$= a^3 - a^2b - ab^2 + b^3.$$

Remark 2. The product of a zero polynomial by any polynomial is, obviously, a zero polynomial.

For example,

$$(a - a)(a^2 + ab + b^2) = 0\,(a^2 + ab + b^2)$$
$$= 0a^2 + 0ab + 0b^2 = 0 + 0 + 0 = 0.$$

SECTION TEST

QUESTIONS

1. What is the rule for multiplying polynomials?
2. How is the factorization of a polynomial into factors defined?

EXERCISES

1. Transform each indicated product of polynomials into a polynomial of a standard form:

(a) $(a + 2b)(a^2 - 2ab + 4b^2)$; (b) $(a + b + c)(a + b - c)$;

(c) $(a + 2b)(a - 2b)(a^2 + 4b^2)$.

Sec. 6.7. Integral Expressions

An *integral expression* is defined as such an algebraic expression in which several polynomials are connected by the signs of addition, subtraction, and multiplication.

It is clear that a polynomial is an integral expression. Here are some more examples of integral expressions:

$$(a + b)(c - d) + (a - b) - (c - d);$$
$$-a(b - c)^3 + (d - c) - a^3 - 5;$$
$$-(a - b) - cd.$$

Integral expressions can be simplified according to the rules obeyed by polynomials and also by applying the rules for addition, subtraction, and multiplication of polynomials.

These rules imply that any *integral expression can be transformed into a standard-form polynomial.*

Consider a couple of examples of simplifying integral expressions.

Example 1.

$$\begin{aligned}(a+2b)(a-2b)(a^2+4b^2)-(a^4-b^4)+(b^4-2a^4)\\ =(aa+2ba-2ab-2b2b)(a^2+4b^2)-(a^4-b^4)+(b^4-2a^4)\\ =(a^2-4b^2)(a^2+4b^2)-(a^4-b^4)+(b^4-2a^4)\\ =(a^2a^2-4b^2a^2+a^24b^2-4b^24b^2)-(a^4-b^4)+(b^4-2a^4)\\ =(a^4-16b^4)-(a^4-b^4)+(b^4-2a^4)\\ =a^4-16b^4-a^4+b^4+b^4-2a^4=-2a^4-14b^4.\end{aligned}$$

Example 2.

$$\begin{aligned}15\alpha^3\beta^2-(3\alpha^2\beta+\alpha)(5\alpha\beta-2)\\ =15\alpha^3\beta^2-3\alpha^2\beta5\alpha\beta+3\alpha^2\beta2-\alpha5\alpha\beta+2\alpha\\ =15\alpha^3\beta^2-15\alpha^3\beta^2+6\alpha^2\beta-5\alpha^2\beta+2\alpha=\alpha^2\beta+2\alpha.\end{aligned}$$

SECTION TEST

QUESTIONS

1. What is an integral expression?
2. Can any integral expression be transformed into a standard-form polynomial?

EXERCISES

1. Simplify the following integral expressions:
(a) $15x^3y^2-(5xy-2)(3x^2y+x)$;
(b) $(a+b+c)(a+b-c)-2ab$;
(c) $(\alpha+2\beta)(\alpha+\gamma)-(\alpha-2\beta)(\alpha-\gamma)$.

Sec. 6.8. Numerical Value of an Integral Expression

First consider an integral expression containing one letter, for instance,

$$a^2+5a-13. \tag{1}$$

If instead of the letter a (no matter where it stands in this expression) we substitute the number 3, we get the numerical expression

$$3\cdot3+5\cdot3-13,$$

equal to the number 11. This number 11 is called the numerical value of Expression (1) for $a = 3$. It is easily seen that the numerical

value of the same expression for $a = 0$ is equal to (-13) and for $a = -0.1$ to (-13.49), and so forth.

Consider now the expression

$$0.2\alpha + 3\beta - \alpha\beta + \frac{7}{4}, \tag{2}$$

containing two letters. If instead of the letter α (no matter what place is occupied by this letter in the expression), we substitute the number (-0.1), and instead of the letter β (regardless of the place occupied by it in this expression) the number 2.5, then we get the numerical expression

$$0.2 \cdot (-0.1) + 3 \cdot 2.5 - (-0.1) \cdot (2.5) + \frac{7}{4},$$

equal to the number 9.48. This number is called the numerical value of Expression (2) for $\alpha = 0.1$ and $\beta = 2.5$.

It is obvious that for $\alpha = 0$ and $\beta = 0$ the numerical value of this expression is equal to $\frac{7}{4}$, for $\alpha = -3$ and $\beta = 0$ to $\frac{23}{20}$, and so on.

The numerical values of integral expressions containing three, four, and more letters are determined in a similar way.

Example 1. The numerical value of the expression

$$\frac{1}{6}a - \frac{1}{15}b + c(a + b)$$

for $a = 3$, $b = -5$, $c = 0.3$ is equal to

$$\frac{1}{6} \cdot 3 - \frac{1}{15} \cdot (-5) + 0.3[3 + (-5)] = \frac{7}{30}.$$

Example 2. The numerical value of the expression

$$x - y + t(z - x) + z(t + y)$$

for $x = -0.1$, $y = -3.2$, $z = 1.7$, $t = 3.5$ is equal to

$$(-0.1) - (-3.2) + 3.5[1.7 - (-0.1)] + 1.7[3.5 + (-3.2)] = 9.91.$$

We should like to underline that for any integral expression for any chosen numerical values of the letters entering this expression the corresponding numerical expression always has sense since it does not contain division by zero.

SECTION TEST

QUESTIONS

1. What is the numerical value of an integral expression for given numerical values of the letters involved?
2. For what numerical values of letters does an integral expression have the corresponding numerical value?

EXERCISES

1. Compute the numerical values (or evaluate) of the following integral expressions:

(a) $a^2 + 5a - 13$ for $a = -3$;

(b) $0.2\alpha^2 + 3\beta - \frac{1}{5}\alpha + \frac{7}{4}$ for $\alpha = 1$, $\beta = -2$;

(c) $x - y + (z - x) + z(t + y)$ for $x = 0$, $y = -1$, $z = -3$, $t = 2$;

(d) $2x + 3y - z + 3$ for $x = 1$, $y = -1$, $z = -1$;

(e) $\frac{1}{3}a - \frac{1}{15}b + c(a + b)$ for $a = 3$, $b = -5$, $c = 0.3$;

(f) $(a - b)(c - d)$ for $a = 1$, $b = 2$, $c = -3$, $d = 4$.

Sec. 6.9. Identical Equality of Integral Expressions

As was already noted, the algebraic equalities considered in Chapters 5 and 6 turn into true numerical equalities if letters in them are replaced by numbers. In other words, the numerical values of both sides of such an equality are equal for any numerical values of the letters contained.

For instance, consider the algebraic equalities

$$x^2 \frac{1}{4} y^2 x = \frac{1}{4} x^3 y^2;$$

$$2x^2 + y^2 - x^2 = x^2 + y^2$$

Each of them turns into a true numerical equality if x and y are replaced by any numbers, for example, $x = 0$ and $y = -1$; $x = -5$ and $y = -7$, and so on.

Therefore we say that the *left-hand side of each of these equalities is identically equal to its right-hand side*, and the equalities themselves are called *identities*.

All the algebraic equalities we considered above are identities, therefore they turn into true numerical equalities when any numbers are substituted for the letters.

Example 1. The algebraic equalities

$$y^2 x^2 2y^2 x = 2x^3 y^4;$$

$$-4ab^2 = (-1)\, 4ab^2$$

are at the same time identities since, by substituting any numbers instead of the letters, we, obviously, get identical numbers on the right and on the left.

Example 2. The algebraic equalities

$$(a + b)(c + d) = ac + ad + bc + bd;$$

$$(a + b) - (a - b) = 2b$$

are identities because they turn into true numerical equalities for any numerical values of the letters entering them.

Finally, note that zero polynomials are identically equal to zero, that is, for any numerical values of the letters entering them their numerical values are equal to zero. Such polynomials may be exemplified by the following:

$$a - a; \quad 3x^2 - x^2 - 2x^2.$$

The remaining (nonzero) polynomials can turn into zero only for certain numerical values, but not identically, that is, for every nonzero polynomial there are numerical values of the letters for which the polynomial does not turn into zero.

We are not going to prove this statement here.

Here are examples of nonzero polynomials:

$$a + b; \quad x - y; \quad \alpha^2 + \beta^2 + 1.$$

The polynomial $a + b$ turns into zero only for the numerical values of a and b satisfying the equality $a = -b$, but for the rest of numerical values of a and b it does not turn into zero.

The polynomial $x - y$ turns into zero only for the numerical values of x and y satisfying the equality $x = y$, but it does not turn into zero for the remaining numerical values of x and y.

The polynomial $\alpha^2 + \beta^2 + 1$ turns into zero for none of the numerical values of α and β.

SECTION TEST

QUESTIONS

1. When are two integral expressions not equal identically?
2. What is an identity?

EXERCISES

1. Are the following pairs of expressions equal identically:
(a) $x + y$ and $y + x$; (b) $\gamma(3\beta\alpha)$ and $3\alpha\beta\gamma$?
2. Are the following equalities identities:
(a) $x + y - 2x + 3y = 4y - x$; (b) $2\alpha - \beta \cdot 3 + 3\beta = 2\alpha$?

Chapter 7

FORMULAS FOR SHORT-CUT MULTIPLICATION OF POLYNOMIALS

Sec. 7.1. The Square of a Binomial: the Square of a Sum

According to the definition of the second power, we have

$$(a + b)^2 = (a + b)(a + b). \tag{1}$$

Applying the rule for multiplying a polynomial by a polynomial, we get

$$(a + b)(a + b) = aa + ba + ab + bb = a^2 + 2ab + b^2. \quad (2)$$

Equalities (1) and (2) yield the equality

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (3)$$

which is called the *formula for the square of the sum.*

Since in Formula (3) we may assume that a and b are arbitrary numbers, it is usually formulated as follows:

The square of the sum of two numbers is equal to the square of the first number plus twice the product of the two numbers, plus the square of the second number.

This formula is frequently used to simplify calculations. For instance, $51^2 = (50 + 1)^2 = 50^2 + 2 \cdot 50 + 1^2 = 2601$.

When read from right to left, Formula (3) shows that the trinomial $a^2 + 2ab + b^2$ can be factorized, namely, it can be represented as the product of two identical factors $(a + b)$.

SECTION TEST

EXERCISES

1. Rewrite the following expressions in the form of a polynomial:
(a) $(a + 2b)(a + 2b)$; (b) $(2x + 3y)^2$; (c) $(3x + y)^2 + (x + 3y)^2$.
2. Find out whether the following polynomials are the squares of certain binomials:
(a) $a^2 + 4ac + 4c^2$; (b) $1 + x^2 + 2x$; (c) $a^2c^2 + 2acd + d^2$.

Sec. 7.2. The Square of a Binomial: the Square of a Difference

The following equalities are obvious:

$$(a - b)^2 = (a - b)(a - b) = aa - ba - ab + bb = a^2 - 2ab + b^2,$$

whence

$$(a - b)^2 = a^2 - 2ab + b^2. \quad (1)$$

This equality is called the *formula for the square of the difference.* It is convenient to be memorized in the following wording:

The square of the difference of two numbers is equal to the square of the first number minus twice the product of the two numbers plus the square of the second number.

This formula is also frequently used for simplification of computations. For instance, $49^2 = (50 - 1)^2 = 50^2 - 2 \cdot 50 + 1^2 = 2401$.

When read from right to left, Formula (1) indicates that the polynomial $a^2 - 2ab + b^2$ can be represented as the product of two identical factors $(a - b)$.

Remark. Formula (1) can be obtained from Formula (3) of Sec. 7.1 by replacing b by $(-b)$.

SECTION TEST

EXERCISES

1. Rewrite the following expressions in the form of a polynomial:
(a) $(x - 2y)^2$; (b) $(\alpha\beta - \gamma)^2$; (c) $(5xy - 2)^2$.
2. Find out whether the following polynomials are the squares of certain binomials:
(a) $a^2 - 4ab + 4b^2$; (b) $x^2 - 4x + 4$; (c) $a^4 - 2a^2 + 1$.

Sec. 7.3. Separating a Perfect Square

Consider several examples.

Example 1. Consider the polynomial of second degree with respect to the letter x:

$$x^2 + 5x + 6.$$

This trinomial can be transformed in the following way:

$$x^2 + 5x + 6 = x^2 + 2 \cdot x \cdot \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6 = \left(x + \frac{5}{2}\right)^2 - \frac{1}{4}.$$

Here, we represented $5x$ as twice the product of $\frac{5}{2}$ and x, added to, and subtracted from, the trinomial one and the same number $\left(\frac{5}{2}\right)^2$, and then applied the formula for the square of the sum to the binomial $\left(x + \frac{5}{2}\right)$.

Thus, we have proved the equality

$$x^2 + 5x + 6 = \left(x + \frac{5}{2}\right)^2 - \frac{1}{4}, \qquad (1)$$

which shows that the second-degree polynomial $x^2 + 5x + 6$ is equal to the perfect square $\left(x + \frac{5}{2}\right)^2$, plus the number $\left(-\frac{1}{4}\right)$. We say, that with the aid of Equality (1), a perfect square is separated from the polynomial $x^2 + 5x + 6$.

Example 2. Consider the second-degree binomial

$$x^2 - 8x.$$

Let us make the following transformations:

$$x^2 - 8x = x^2 - 2 \cdot x \cdot 4 + 4^2 - 4^2 = (x - 4)^2 - 16.$$

Here, we represented $8x$ as twice the product of x and 4, added to, and subtracted from, the polynomial one and the same number 4^2, and, finally, applied the formula for the square of the difference to the binomial $(x - 4)$. Thus, we have proved the equality

$$x^2 - 8x = (x-4)^2 - 16, \qquad (2)$$

which shows that the second-degree polynomial $x^2 - 8x$ is equal to the perfect square $(x - 4)^2$ plus the number (-16).

We say, that with the aid of Equality (2), a perfect square is separated from the polynomial $x^2 - 8x$.

Reasoning in a similar way, we can separate a perfect square from any second-degree polynomial with respect to x with coefficient of x^2 equal to 1.

Example 3.

$$x^2 + x + 1 = x^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Example 4.

$$x^2 - 7x + 19 = x^2 - 2 \cdot x \cdot \frac{7}{2} + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 19 = \left(x - \frac{7}{2}\right)^2 + \frac{27}{4}.$$

If a second-degree polynomial is the square of a certain binomial, then, applying to it the method of separating a perfect square, we just obtain the formula for the square of this binomial.

Example 5.

$$x^2 + 4x + 4 = x^2 + 2 \cdot x \cdot 2 + 2^2 - 2^2 + 4 = (x + 2)^2.$$

Example 6.

$$x^2 - 6x + 9 = x^2 - 2 \cdot x \cdot 3 + 3^2 - 3^2 + 9 = (x - 3)^2.$$

SECTION TEST

QUESTIONS

1. What is meant by separating a perfect square from a second-degree polynomial?

EXERCISES

1. Separate a perfect square from the following second-degree polynomials: (a) $x^2 - 4x + 5$; (b) $x^2 + 3x - 1$; (c) $x^2 - 5x$.

Sec. 7.4. The Cube of a Binomial: the Cube of a Sum

By the property of a natural power, we have

$$(a + b)^3 = (a + b)^2 (a + b).$$

Applying now the formula for the square of the sum, the rule for multiplying polynomials, and collecting like terms, we get

$$(a + b)^3 = (a + b)^2 (a + b) = (a^2 + 2ab + b^2)(a + b)$$
$$= a^3 + 2a^2b + b^2a + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Thus, we have proved the equality

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (1)$$

which is called the *formula for the cube of the sum*. It is convenient to be memorized in the following formulation:

The cube of the sum of two numbers is equal to the cube of the first number, plus three times the product of the square of the first number by the second, plus three times the product of the first number by the square of the second, plus the cube of the second number.

When read from right to left, Formula (1) indicates that the polynomial $a^3 + 3a^2b + 3ab^2 + b^3$ can be factorized, namely, it can be represented as the product of three like factors $(a + b)$.

SECTION TEST

EXERCISES

1. Write the following expressions in the form of a polynomial:
(a) $(x + 3z)^3$; (b) $(a + 2)^3$; (c) $(2b + 3)^3$.
2. Find out whether the following polynomials are the cubes of certain binomials:
(a) $8x^3 + 12x^2y + 6xy^2 + y^3$; (b) $\alpha^3 + 3\alpha^2 + 3\alpha + 1$;
(c) $27 + 27b + 9b^2 + b^3$.

Sec. 7.5. The Cube of a Binomial: the Cube of a Difference

Reasoning as in the preceding section, we get

$$(a - b)^3 = (a^2 - 2ab + b^2)(a - b)$$
$$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

that is, we get the equality

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3. \qquad (1)$$

This equality is called the *formula for the cube of the difference*. It is convenient to be memorized in the following wording:

The cube of the difference of two numbers is equal to the cube of the first number, minus three times the product of the square of the first number by the second, plus three times the product of the first number by the square of the second, minus the cube of the second number.

Rewriting Formula (1) in the form

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3,$$

we get the polynomial $a^3 - 3a^2b + 3ab^2 - b^3$ factorized into factors.

SECTION TEST

EXERCISES

1. Write the following expression in the form of a polynomial:
(a) $(x - 1)^3$; (b) $(2a - 3b)^3$; (c) $(a^2 - 2b)^3$.
2. Find out whether the following polynomials are the cubes of certain binomials:
(a) $1 - 3x + 3x^2 - x^3$; (b) $a^3 - 6a^2 + 12a - 8$;
(c) $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

Sec. 7.6. The Difference Between Two Squares

Consider the product

$$(a + b)(a - b).$$

Applying the rule for multiplication of polynomials and collecting like terms, we get

$$(a + b)(a - b) = a^2 + ab - ab - b^2 = a^2 - b^2.$$

Thus, we have proved the equality

$$a^2 - b^2 = (a + b)(a - b), \qquad (1)$$

which is called the *formula for the difference between two squares.*

It is convenient to be memorized in the following formulation:

The difference between the squares of two numbers is equal to the product of the sum and difference of these numbers.

Formula (1) shows that the polynomial $a^2 - b^2$ has been factored into two binomial factors.

SECTION TEST

EXERCISES

1. Write the following products in the form of a polynomial:
(a) $(x - 2)(x + 2)$; (b) $(a^2 + 1)(a^2 - 1)$;
(c) $(2\alpha - 3\beta)(2\alpha + 3\beta)$.
2. Factor the following polynomials:
(a) $4a^2 - 1$; (b) $a^2 - b^2$; (c) $9x^4 - 4$; (d) $x^4 - 16$.

Sec. 7.7. The Sum of Two Cubes

Applying the rule for multiplication of polynomials and collecting like terms, we have

$$(a + b)(a^2 - ab + b^2) = a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3$$
$$= a^3 + b^3.$$

Thus, we have proved the equality

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2). \quad (1)$$

This equality is called the *formula for the sum of two cubes.*

The polynomial $a^2 - ab + b^2$ is called the *imperfect square of the difference between a and b.*

Formula (1) is convenient to be memorized in the following formulation:

The sum of the cubes of two numbers is equal to the product of the sum of these numbers by the imperfect square of the difference between these numbers.

Formula (1) yields the factorization of the polynomial $a^3 + b^3$ into two factors.

SECTION TEST

EXERCISES

1. Write each of the following products in the form of a polynomial:
(a) $(3x + y)(9x^2 - 3xy + y^2)$; (b) $(a^4 + 1)(a^8 - a^4 + 1)$;
(c) $(4\alpha^2 - 6\alpha + 9)(2\alpha + 3)$.
2. Factor the following polynomials:
(a) $a^6 + b^3$; (b) $27\alpha^3 + 8\beta^3$.

Sec. 7.8. The Difference Between Two Cubes

Reasoning as in the preceding section, we get

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \quad (1)$$

This equality is called the *formula for the difference of two cubes.*

The polynomial $a^2 + ab + b^2$ is called the *imperfect square of the sum of a and b.*

Formula (1) is convenient for memorizing in the following formulation:

The difference of the cubes of two numbers is equal to the product of the difference between these numbers by the imperfect square of the sum of these numbers.

Formula (1) gives the binomial $a^3 - b^3$ factored into two polynomial factors.

SECTION TEST

EXERCISES

1. Write each of the following products in the form of a polynomial:
(a) $(2a - 3b)(4a^2 + 6ab + 9b^2)$; (b) $(a^2 - 1)(a^4 + a^2 + 1)$;
(c) $(x^2y^2z^2 + xyzt + t^2)(xyz - t)$.
2. Factor the following polynomials:
(a) $a^3 - 8b^3$; (b) $8a^6 - 27$; (c) $x^3y^3 - z^6$.

Sec. 7.9. Applying the Formulas for Short-cut Multiplication

The formulas proved in the present chapter remain true if a and b are replaced in them by any integral expressions.

For instance, the expression

$$(a+b+c+d)^2$$

may be considered as the square of the sum of two integral expressions $(a + b)$ and $(c + d)$, therefore the following equalities hold true:

$$\begin{aligned}(a+b+c+d)^2 &= [(a+b)+(c+d)]^2\\ &= (a+b)^2+2(a+b)(c+d)+(c+d)^2\\ &= a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2\\ &= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.\end{aligned}$$

Thus, we have obtained the formula for the square of the sum of four numbers:

$$\begin{aligned}&(a + b + c + d)^2\\ &\quad = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.\end{aligned}$$

The formulas for short-cut multiplication are frequently used for simplification of integral expressions. For instance,

$$\begin{aligned}&(a + 1)(a - 1)(a^4 + a^2 + 1) + (a^2 - a + 1)(a + 1)\\ &\qquad = (a^2 - 1)(a^4 + a^2 + 1) + (a + 1)(a^2 - a + 1)\\ &\qquad = (a^6 - 1) + (a^3 + 1) = a^6 + a^3.\end{aligned}$$

Here, we applied in succession the formulas for the difference between two squares, for the difference and the sum of two cubes. Also, we should like to note here that the formulas for short-cut multiplication are used to factor polynomials into factors. This will be considered in the next section.

SECTION TEST

EXERCISES

1. Write the following expressions in the form of a polynomial:
(a) $(a + 1)(a + 2)(a^2 + 4)(a^2 + 1)(a - 2)(a - 1)$;
(b) $(a + b + c)(a + b - c) - 2ab$;
(c) $(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)$.
2. Factor the following expressions:
(a) $y^2 - 10y + 25 - 4x^2$; (b) $(a + b)^3 - a^3 - b^3$; (c) $x^{16} - y^{16}$.

Sec. 7.10. Factorization of Polynomials

To factor a polynomial into factors (we mean nontrivial factors that is, those having a positive exponent) is a difficult problem, by far not always solvable. There are polynomials which are not factorable at all, the binomial $a^2 + b^2$ being an example of such polynomials. On the other hand, a polynomial can be factored into factors, but we do not know the rules indicating how this factorization is realized. That is why of high value are those cases when such rules are at our disposal.

From this point of view, of great importance are the formulas derived in the previous sections, that is, the formulas for the square of the sum and difference of two numbers, the cube of the sum and difference, the difference between two squares, the difference and sum of two cubes—all of them yield factorization of important polynomials into polynomial factors.

If all terms of a polynomial contain the same monomial as a factor, then it can be taken outside the brackets. This technique is also valuable and is frequently used in factoring polynomials.

It is sometimes possible to carry out factoring by various methods of grouping. This is a more difficult technique requiring practical habits and quick wits. It consists in breaking the terms of a polynomial up into several groups, taking out a factor in each group, and finding the same expression in all sets of brackets. This expression can then be factored out.

Factoring by grouping of terms is shown here by examples.

Example 1.

$$2ax - 2ay + 3by - 3bx = (2ax - 2ay) + (3by - 3bx)$$
$$= 2a(x - y) + 3b(y - x) = 2a(x - y) - 3b(x - y) = (x - y)(2a - 3b).$$

Example 2.

$$a^4 - ab^3 - a^2b^2 + b^4$$
$$= (a^4 - ab^3) - (a^2b^2 - b^4) = a(a^3 - b^3) - b^2(a^2 - b^2)$$
$$= a(a - b)(a^2 + ab + b^2) - b^2(a - b)(a + b)$$
$$= (a - b)(a^3 + a^2b + ab^2) - (a - b)(ab^2 + b^3)$$
$$= (a - b)(a^3 + a^2b + ab^2 - ab^2 - b^3) = (a - b)(a^3 + a^2b - b^3).$$

Note, that sometimes, a polynomial can be factored into factors by making use of the method of separating a perfect square and applying then the formula for the difference between two squares.

Example 3.

$$x^2 + 2x - 15 = x^2 + 2 \cdot x \cdot 1 + 1^2 - 1^2 - 15 = (x + 1)^2 - 16$$
$$= (x + 1)^2 - 4^2 = [(x + 1) + 4]\,[(x + 1) - 4]$$
$$= (x + 5)\,(x - 3).$$

Example 4.

$$x^4 - 10x^2 + 9 = (x^2)^2 - 2 \cdot x^2 \cdot 5 + 5^2 - 5^2 + 9 = (x^2 - 5)^2 - 16$$
$$= (x^2 - 5)^2 - 4^2 = [(x^2 - 5) + 4]\,[(x^2 - 5) - 4]$$
$$= (x^2 - 1)\,(x^2 - 9) = (x - 1)\,(x + 1)\,(x - 3)(x + 3).$$

SECTION TEST

QUESTIONS

1. What methods (familiar to you) can be applied for factoring a polynomial?

EXERCISES

1. Factor the following polynomials:
(a) $ab + cb + ad + cd$; (b) $a^2 - 2ab + b^2 - c^2$;
(c) $a^4 - 16b^4$; (d) $a^2 + 2ab + ac + b^2 + bc$;
(e) $9y^2 - 6y + 1 - x^2$; (f) $x^4 + 4x^2 - y^2 + 6y - 5$.

Chapter 8

ALGEBRAIC FRACTIONS

In Chapter 6 we studied polynomials. In particular, we showed that the sum, difference, and product of two polynomials is a polynomial. Let us now consider the quotient of two polynomials.

Sec. 8.1. The Notion of an Algebraic Fraction

We shall denote polynomials by the letters A, B, C. An *algebraic fraction* is defined as the literal expression $\frac{A}{B}$ which is the quotient from the division of the polynomial A by a nonzero polynomial B. The polynomial A is called the *numerator* of the algebraic fraction $\frac{A}{B}$, and the polynomial B the *denominator* of this fraction. We recall once again that it is always supposed that the denominator B of the algebraic fraction A is a nonzero polynomial. Therefore, further on, when we shall say that we are given a fraction $\frac{A}{B}$, we shall take it for granted that B is a nonzero polynomial.

The following expressions may serve as examples of algebraic fractions:

$$\frac{a^2-b^2}{a^2+b^2+1};\quad \frac{a^2+b^2}{a-b};\quad \frac{a-b}{a^3-b^3};\quad \frac{\alpha^2+\alpha\beta+\beta^2}{3}.$$

The denominator of the last fraction is the number 3. The reader should remember that any number may be regarded as a polynomial.

If the polynomial A is a number α, that is, $A = \alpha$, and the nonzero polynomial B is a number β, that is, $B = \beta$ and $\beta \neq 0$, then the quotient $\frac{A}{B}$ is understood as an ordinary numerical fraction $\frac{\alpha}{\beta}$ that is, as a number $\frac{\alpha}{\beta}$:

$$\frac{A}{B} = \frac{\alpha}{\beta},$$

which may also be regarded as a polynomial.

For instance, if $A = -5$ and $B = 3$, then the algebraic fraction $\frac{A}{B}$ is actually the number $\left(-\frac{5}{3}\right)$.

By definition, the following algebraic equalities take place:

(1) $\frac{A}{1} = A$,

that is, the division of a polynomial by unity yields the same polynomial. This rule states that any polynomial A may be regarded as an algebraic fraction $\frac{A}{1}$.

(2) $\frac{AC}{BC} = \frac{A}{B}$

for any nonzero polynomial C.

This is the *fundamental property of algebraic fractions*. It is usually formulated as follows:

An algebraic fraction may be reduced by a nonzero polynomial.

For instance,

$$\frac{(a+b)(a^2+b^2)}{a^2-b^2} = \frac{(a^2+b^2)(a+b)}{(a-b)(a+b)} = \frac{a^2+b^2}{a-b};$$

$$\frac{a^3-b^3}{a^2-b^2} = \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)} = \frac{a^2+ab+b^2}{a+b}.$$

(3) $-\frac{A}{B} = \frac{-A}{B} = \frac{A}{-B}$,

that is, if $\frac{A}{B}$ is an algebraic fraction, then the expression $\left(-\frac{A}{B}\right)$ is also an algebraic fraction, namely, the quotient from the division of the polynomial $(-A)$ by the polynomial B or of the polynomial A by the polynomial $(-B)$.

For instance,

$$-\frac{x^2-xy+y^2}{x-y}=\frac{xy-x^2-y^2}{x-y}=\frac{x^2-xy+y^2}{y-x}.$$

SECTION TEST

QUESTIONS

1. What is an algebraic fraction?
2. What rules do algebraic fractions obey?
3. What is the fundamental property of an algebraic fraction?

EXERCISES

1. Reduce the following fractions:

(a) $\frac{a^2-b^2}{a+b}$; (b) $\frac{a^3+b^3}{a+b}$; (c) $\frac{(2a+b)(\alpha-\beta)}{2a+b}$.

Sec. 8.2. Arithmetic Operations with Algebraic Fractions

The fractions $\frac{A}{B}$ and $\frac{A_1}{B_1}$ are added, subtracted, multiplied, and divided according to the following rules:

(1) $\frac{A}{B}+\frac{A_1}{B_1}=\frac{AB_1+BA_1}{BB_1}$;

(2) $\frac{A}{B}-\frac{A_1}{B_1}=\frac{AB_1-A_1B}{BB_1}$;

(3) $\frac{A}{B}\cdot\frac{A_1}{B_1}=\frac{AA_1}{BB_1}$;

(4) $\frac{A}{B}\div\frac{A_1}{B_1}=\frac{AB_1}{BA_1}$.

In other words, we shall regard that the algebraic equalities (1), (2), (3) and (4) are true. Rule (4) is applied, naturally, provided that A_1 is a nonzero polynomial (the polynomials B and B_1 were already assumed to be nonzero).

Example 1.

$$\frac{1}{x-a}+\frac{1}{x+a}=\frac{x+a+x-a}{(x-a)(x+a)}=\frac{2x}{x^2-a^2}.$$

Example 2.

$$\frac{1}{x+a}+\frac{a-x}{x^2-ax+a^2}=\frac{x^2-ax+a^2+(a+x)(a-x)}{(x+a)(x^2-ax+a^2)}$$

$$=\frac{x^2-ax+a^2+a^2-x^2}{x^3+a^3}=\frac{2a^2-ax}{x^3+a^3}=\frac{a(2a-x)}{x^3+a^3}.$$

Example 3.

$$\frac{1}{x^2-a^2}-\frac{1}{x^2+a^2}=\frac{(x^2+a^2)-(x^2-a^2)}{x^4-a^4}=\frac{2a^2}{x^4-a^4}.$$

Example 4.

$$\frac{a^3-b^3}{a+b}\cdot(a^2-b^2)=\frac{a^3-b^3}{a+b}\cdot\frac{a^2-b^2}{1}$$
$$=\frac{(a^3-b^3)(a^2-b^2)}{(a+b)\cdot 1}=\frac{(a^3-b^3)(a-b)}{1}=(a^3-b^3)(a-b).$$

We may omit the intermediate work here since it is obvious and write at once:

$$\frac{a^3-b^3}{a+b}\cdot(a^2-b^2)=(a^3-b^3)(a-b).$$

Example 5.

$$\frac{a+b}{a-b}\div\frac{(a+b)^2}{a^2-b^2}=\frac{(a+b)(a^2-b^2)}{(a-b)(a+b)^2}=\frac{(a+b)(a-b)(a+b)}{(a-b)(a+b)^2}=1.$$

Note that for $B=B_1$ the rule for addition may be replaced by a simpler one:

$$\frac{A}{B}+\frac{A_1}{B}=\frac{A+A_1}{B},$$

since

$$\frac{A}{B}+\frac{A_1}{B}=\frac{AB+A_1B}{B^2}=\frac{(A+A_1)B}{B^2}=\frac{A+A_1}{B}.$$

For example,

$$\frac{a}{x^2+ax+a^2}+\frac{x}{x^2+ax+a^2}=\frac{a+x}{x^2+ax+a^2}.$$

But it may also happen that the polynomials B and B_1 have a common factor:

$$B=PQ,\quad B_1=PQ_1,$$

where P, Q, Q_1 are polynomials. Then computational work will be less complicated if we shall reason in the following way:

$$\frac{A}{B}+\frac{A_1}{B_1}=\frac{A}{PQ}+\frac{A_1}{PQ_1}=\frac{AQ_1}{PQQ_1}+\frac{A_1Q}{PQQ_1}=\frac{AQ_1+A_1Q}{PQQ_1}.$$

Similar notes may also be mentioned concerning the rule for subtraction.

This is a well-known scheme which is used in arithmetic to simplify the addition of rational numbers. In arithmetic the letters A, A_1,

B, B_1, P, Q, Q_1 denote whole numbers, while in our case these are polynomials.

For instance,

$$\frac{1}{(x+a)^2(x-a)}+\frac{1}{(x^2-a^2)(x-a)}=\frac{1}{(x+a)(x^2-a^2)}+\frac{1}{(x-a)(x^2-a^2)}$$
$$=\frac{x-a}{(x^2-a^2)(x+a)(x-a)}+\frac{x+a}{(x^2-a^2)(x-a)(x+a)}$$
$$=\frac{(x-a)+(x+a)}{(x^2-a^2)(x^2-a^2)}=\frac{2x}{(x^2-a^2)^2}.$$

SECTION TEST

QUESTIONS

1. What are the rules for performing arithmetic operations with algebraic fractions?

2. How is the rule for subtracting the fractions $\frac{A}{B}$ and $\frac{A_1}{B_1}$ formulated if $B=PQ$ and $B_1=PQ_1$?

EXERCISES

1. Find the sum and difference of the following algebraic fractions:

(a) $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$, (b) $\frac{\alpha-\beta}{2}$ and $\frac{\alpha^3+\beta^3}{\alpha^2}$.

2. Find:

(a) the product of the fractions: $\frac{a^2-b^2}{a}$ and $\frac{b}{a+b}$; (b) the quotient of the fractions $\frac{a+b}{a^3-b^3}$ and $\frac{a^3+b^3}{a-b}$.

Sec. 8.3. Properties of Algebraic Fractions

The rules set forth in the previous two sections imply a number of properties of algebraic fractions. Some of them are given below.

1. If B is a nonzero polynomial, then

$$\frac{0}{B}=0.$$

Indeed,

$$\frac{0}{B}=\frac{0\cdot B}{1\cdot B}=\frac{0}{1}=0.$$

The proof, as we see, is very simple. We have the right to replace the numerator 0 by $0\cdot B$ and the denominator B by $1\cdot B$ since $0\cdot B = 0$ and $1\cdot B = B$. Further, taking advantage of the fact that B is a nonzero polynomial, we have the right, on the basis of the funda-

mental property of algebraic fractions, to reduce the obtained fraction by B. As a result, we get $\frac{0}{1}$ which is equal to the number 0 (that is, to a zero polynomial).

For instance,

$$\frac{0}{x-y}=0,$$

since $x-y$ is a nonzero polynomial.

2. $\frac{A}{B}=A\cdot\frac{1}{B}$.

Indeed,

$$\frac{A}{B}=\frac{A\cdot 1}{1\cdot B}=\frac{A}{1}\cdot\frac{1}{B}=A\cdot\frac{1}{B}.$$

3. $\frac{1}{AB}=\frac{1}{A}\cdot\frac{1}{B}$.

Indeed,

$$\frac{1}{AB}=\frac{1\cdot 1}{A\cdot B}=\frac{1}{A}\cdot\frac{1}{B}.$$

4. $\frac{A}{7}=\frac{1}{7}\cdot A$.

Indeed,

$$\frac{A}{7}=\frac{1\cdot A}{7\cdot 1}=\frac{1}{7}\cdot\frac{A}{1}=\frac{1}{7}\cdot A.$$

Hence, the fraction $\frac{A}{7}$ may also be regarded as the polynomial $\frac{1}{7}\cdot A$. Of course, in this example the number 7 may be replaced by any other nonzero number.

Suppose we are given two algebraic fractions $\frac{A}{B}$ and $\frac{A_1}{B_1}$. Multiplying the numerator and denominator of the first fraction by B_1, and the numerator and denominator of the second fraction by B, we get (on the basis of the fundamental property of algebraic fractions)

$$\frac{A}{B}=\frac{AB_1}{BB_1},\quad \frac{A_1}{B_1}=\frac{A_1B}{BB_1}.$$

We see that the fractions $\frac{AB_1}{BB_1}$ and $\frac{A_1B}{BB_1}$ have the same denominator. In such cases we say that we have reduced the fractions $\frac{A}{B}$ and $\frac{A_1}{B_1}$ to a common denominator.

Example. Reduce the fractions

$$\frac{a}{a-b}\text{ and }\frac{b}{a+b}$$

to a common denominator:

$$\frac{a}{a-b}=\frac{a(a+b)}{a^2-b^2};\quad \frac{b}{a+b}=\frac{b(a-b)}{a^2-b^2}.$$

As the rule for adding algebraic fractions shows, they are added being reduced to a common denominator

$$\frac{A}{B}+\frac{A_1}{B_1}=\frac{AB_1}{BB_1}+\frac{A_1B}{BB_1}=\frac{AB_1+A_1B}{BB_1}.$$

When adding the fractions $\frac{A}{B}$ and $\frac{A_1}{B_1}$ in the case $B=PQ$ and $B_1=PQ_1$ we also reduced these fractions to a common denominator.

SECTION TEST

QUESTIONS

1. What properties do algebraic fractions possess?
2. What does it mean to reduce algebraic fractions to a common denominator?

EXERCISES

1. Reduce the following algebraic fractions to a common denominator:
(a) $\frac{a}{b}$ and $\frac{b}{a-b}$, (b) $\frac{\alpha+\beta}{\alpha-\beta}$ and $\frac{\alpha^3}{\alpha^2-\beta^2}$.

Sec. 8.4. Rational Expressions

Making use of the rules and properties set forth in previous sections of this chapter, it is possible to simplify expressions containing letters.

Example 1. Let us simplify the expression

$$\frac{1+\frac{1}{a}}{\frac{6}{b}+\frac{3}{a}+\frac{3}{ab}}-\frac{\frac{ab}{3}}{2a+b+1}$$

$$=\frac{\frac{a+1}{a}}{\frac{6a+3b+3}{ab}}-\frac{\frac{ab}{3}}{2a+b+1}$$

$$=\frac{(a+1)ab}{a(6a+3b+3)}-\frac{ab}{3(2a+b+1)}$$

$$=\frac{(a+1)b}{6a+3b+3}-\frac{ab}{6a+3b+3}$$

$$=\frac{ab+b-ab}{6a+3b+3}=\frac{b}{6a+3b+3}$$

by the rule for adding algebraic fractions

by the rule for division of algebraic fractions

on reducing by a nonzero polynomial

by the rule for subtracting algebraic fractions

on collecting like terms.

Rational expression is defined as such an expression in which several algebraic fractions are connected by the signs of arithmetic operations and which does not involve the division by a zero polynomial.

It is clear that an algebraic fraction is a particular case of a rational expression. Here are some other examples of rational expressions:

$$\frac{z+a}{(c+d)^2}+1; \quad \frac{a}{b}-5; \quad \frac{a(b-1)^3+\frac{1}{a}}{d+5\cdot\frac{a}{c}}.$$

Rational expressions can be simplified with the aid of the rules obeyed by algebraic fractions. As is obvious, Example 1 is an example of simplifying a rational expression.

Example 2.

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left[(x-y)^2+xy\right]+\left(\frac{1}{x}-\frac{1}{y}\right)\left[(x+y)^2-xy\right]$$
$$=\frac{x+y}{xy}(x^2-xy+y^2)+\frac{y-x}{xy}(x^2+xy+y^2)$$
$$=\frac{(x+y)(x^2-xy+y^2)}{xy}-\frac{(x-y)(x^2+xy+y^2)}{xy}$$
$$=\frac{x^3+y^3}{xy}-\frac{x^3-y^3}{xy}=\frac{x^3+y^3-x^3+y^3}{xy}=\frac{2y^3}{xy}=2\cdot\frac{y^2}{x}.$$

Example 3.

$$\frac{a^2}{xy}+\frac{(a+x)^2}{x^2-xy}-\frac{(a+y)^2}{yx-y^2}=\frac{a^2}{xy}+\frac{(a+x)^2}{x(x-y)}-\frac{(a+y)^2}{y(x-y)}$$
$$=\frac{a^2(x-y)+(a+x)^2y-(a+y)^2x}{xy(x-y)}$$
$$=\frac{a^2x-a^2y+a^2y+2axy+x^2y-a^2x-2ayx-y^2x}{xy(x-y)}=\frac{x^2y-y^2x}{xy(x-y)}=\frac{xy(x-y)}{xy(x-y)}=1.$$

Remark. We leave out of consideration such expressions which involve division by a zero polynomial since they have no sense.

For example, the expressions

$$\frac{a+b}{\frac{1}{a}-\frac{1}{a}}; \quad \frac{\frac{1}{a}}{c^3-c^3}; \quad \frac{1+\frac{x^2+y^2}{4y^2-(y+y)^2}}{x^2-1}$$

have no sense since they involve division by a zero polynomial.

Here we underline once again that such expressions were not studied before and will not be considered in future.

SECTION TEST

QUESTIONS

1. What is meant by a rational expression?
2. What are the rules for simplifying rational expressions?
3. What is an expression having no sense?

EXERCISES

1. Simplify the following rational expressions:

(a) $\left(\frac{a^3-b^3}{a-b}\right)\left(\frac{a}{a^2+ab+b^2}\right)-(a-b)$;

(b) $\frac{\alpha\beta}{\alpha^3-\beta^3}\div\frac{\alpha+\beta}{\alpha^3-\beta^3}+\frac{\alpha^2}{\alpha+\beta}$.

2. Which of the following expressions have no sense:

(a) $\frac{x-y}{x^2-y^2}$; (b) $\frac{7+\frac{x-a}{a^2-2a^2+a^2}}{x^2+a^2}$;

(c) $\frac{\alpha^2+\beta^2-2\alpha\beta}{(x-5)^2-x^2-25+10x}$; (d) $\frac{1}{a-\frac{1}{a}-\frac{a^2-1}{a}}$?

Sec. 8.5. A Numerical Value of a Rational Expression

Consider, as an example, the rational expression

$$\frac{a^2+1}{a-1}+2a \tag{1}$$

containing one letter.

Substituting the number 3 for a, we get the numerical expression

$$\frac{3^2+1}{3-1}+2\cdot 3,$$

equal to the number 11. The number 11 is called the numerical value or, simply, the value of Expression (1) for $a = 3$.

For $a = -1$ the value of the same expression is equal to

$$\frac{(-1)^2+1}{(-1)-1}+2\cdot(-1)=-3.$$

Proceeding in such a way, we can compute the value (or evaluate) of Expression (1) for any values of the letter a, except for $a = 1$, since for $a = 1$ Expression (1) has no sense:

$$\frac{1^2+1}{1-1}+2\cdot 1,$$

because the denominator turns out to be equal to zero.

Expression (1) *is said to be defined for all numerical values of* a, *except for* $a = 1$.

Consider now, as another example, the rational expression

$$\frac{x^2+y^2}{x-y} \tag{2}$$

containing two letters.

Let us assign two numbers and substitute the first of them into Expression (2) for x, the second for y. If the numerical value of the denominator will turn out to be nonzero, then we obtain a numerical expression equal to a certain number. This number is called the numerical value or, simply, the value of Fraction (2) for the given numerical values of x and y. We see that Fraction (2) has numerical values for any values of x and y if only they differ from each other.

Fraction (2) *is said to be defined for all numerical values of* x *and* y *not equal to each other.*

Draw attention to the fact that the denominator of Fraction (2) is a nonzero polynomial. But for equal values of x and y its value becomes equal to zero. But there are many pairs of numerical values of x and y for which the denominator does not become zero. For each of such pairs our fraction has a numerical value, that is, it is defined.

Numerical values of any rational expressions are determined in a similar way.

For instance, the expression

$$\frac{x+y+z}{x^2+y^2+z^2} - xyz$$

is defined for all numerical values of x, y, and z except for $x = y = z = 0$.

The expression

$$\frac{1+\dfrac{a+a}{(c-d)^2}}{a^2+b^2+1}$$

is defined for all numerical values of a, b, c, d, except for those for which $c = d$.

The algebraic fraction $\frac{A}{B}$ is defined for all numerical values of its letters with the exception of those for which the denominator B becomes zero.

For example, the fraction

$$\frac{x-y-z-t}{2x-3y}$$

is defined for all numerical values of x, y, z, and t except for those for which $2x - 3y = 0$.

Remark. Above, we left out of consideration, as having no sense, such expressions which involve division by a zero polynomial. Such expressions are not defined for any numerical values of their letters.

SECTION TEST

QUESTIONS

1. For what numerical values of the letters is a given rational expression not defined?
2. What is a numerical value of a rational expression?

EXERCISES

1. Find for what numerical values of the letters is each of the following expressions defined:

(a) $\frac{a+b}{a^3-b^3}+\frac{b}{a}$, (b) $\frac{(xy-5)}{(x+y)}\cdot\frac{(x-y)}{xy}$, (c) $\frac{\frac{1}{\alpha}-\frac{1}{\beta}}{\alpha-\beta}$.

2. Find the numerical values of the following expressions:

(a) $\frac{a+b}{a^3-b^3}+a+\frac{b}{a}$ for $a=3$, $b=4$,

(b) $\frac{\alpha\beta}{\alpha^3+\beta^2}-\alpha^3$ for $\alpha=-3$, $\beta=4$,

(c) $\frac{xy-5}{x+y}\cdot\frac{x+y}{x-y}$ for $x=0$, $y=-3$.

Sec. 8.6. Identical Equality of Rational Expressions

The first four sections of the present chapter dealt with algebraic equalities of rational expressions. Here is one of such algebraic equalities:

$$\frac{a^2+a+1}{a-3}=\frac{(a^2+a+1)(a-1)}{(a-3)(a-1)}. \qquad (1)$$

Its left-hand side is defined for all numerical values of a different from 3. And its right-hand side is defined for all numerical values of a different from 3 and 1. But then both sides of Equality (1) are defined for all numerical values of a different from 3 and 1, and for each of the values of a ($a\neq 3$, $a\neq 1$) the numerical values of the left-hand and right-hand sides in the algebraic equality (1) are equal to each other.

This takes place for any algebraic equalities between rational expressions.

An algebraic equality between two rational expressions turns into a true numerical equality for all numerical values of the letters for which both of these expressions are defined.

Here are some more examples of algebraic equalities:

$$\frac{1}{x-2}+\frac{1}{x-3}=\frac{2x-5}{(x-2)(x-3)}; \tag{2}$$

$$\frac{1}{(a-2)(a+1)}\div\frac{1}{a-2}=\frac{1}{a+1}. \tag{3}$$

The algebraic equality (2) turns into a true numerical equality for all numerical values of x different from 2 and 3.

The algebraic equality (3) turns into a true numerical equality for all numerical values of x different from (—1) and 2.

In general, any algebraic equality between two rational expressions turns into a true numerical equality for all those numerical values of its letters for which these expressions are defined. An equality between literal expressions is said to be an *identity* if it turns into a true numerical equality for all those numerical values of their letters for which these expressions are defined.

Thus, *any algebraic equality is also an identity.*

For instance, the algebraic equalities (2) and (3) are identities.

SECTION TEST

QUESTIONS

1. On what set does an algebraic equality turn into an identity?
2. When are two expressions identically equal?

EXERCISES

1. Write out the sets on which the following algebraic equalities are identities:

(a) $\frac{a+b}{a^2-b^2}=\frac{1}{a-b}$; (b) $\frac{a}{a-b}-\frac{b}{a+b}=\frac{a^2+b^2}{a^2-b^2}$.

Chapter 9

LINEAR EQUATIONS IN ONE UNKNOWN

Sec. 9.1. First-degree Equations in One Unknown

The following equations

$$5x-3=0; \quad x+5=0; \quad -2x-7=0; \quad 3x=0$$

may serve as examples of first-degree equations in one unknown.

The expression written to the left of the equality sign is called the *left-hand side* (member) of the equation, and the expression written to the right of the equality sign the *right-hand side* (member).

Now we may say that an *equation of the first degree in one unknown* x is defined as an equation the left-hand side of which is a standard-

form polynomial of the first degree with respect to x, and the right-hand side is zero.

Of course, in this definition the unknown may also be denoted by another letter: y, z, t, ...

The *general equation of the first degree* with the unknown x has the form

$$kx + b = 0,$$

where k and b are given numbers ($k \neq 0$). The number k is called the *coefficient* of the unknown in this equation, and b the *constant term* of this equation. Thus, in the equation

$$5x - 3 = 0$$

5 is the coefficient of the unknown, and (—3) the constant term; in the equation

$$3x = 0$$

3 is the coefficient of the unknown, and 0 the constant term.

The *root* (or *solution*) of an equation is defined as any number which, when substituted for the unknown x, turn the equation into a true numerical equality.

To solve an equation means to find all of its roots. Consider, for instance, the equation

$$2x + 6 = 0. \tag{1}$$

To solve this equation, we may reason in the following way. Suppose Equation (1) has a root. Let us denote it by x_0 and substitute into Equation (1). We get a true numerical equality

$$2x_0 + 6 = 0.$$

Let us add the number (—6) to both sides of this numerical equality or, which is the same, let us transpose the number 6 to the right-hand side of Equation (1). By doing so, we again obtain a true numerical equality

$$2x_0 = -6.$$

Let us now divide this equality by 2, that is, divide both sides of the equality by 2. We then get

$$x_0 = -3.$$

Thus, we have shown that *if Equation* (1) *has a root, then it is equal to the number* (—3).

Let us now check to see that the number (—3) is actually the root of Equation (1). To this end, we substitute it into this equation for the unknown x. We then obtain

$$2 \cdot (-3) + 6 = 0.$$

We see that this is a true numerical equality. Consequently, Equation (1) has the root

$$x_0 = -3$$

which is the only one.

In order to solve the general equation of the first degree

$$kx + b = 0 \tag{2}$$

we shall proceed as above.

Suppose the number x_0 is the root of Equation (2). Substituting it into this equation, we get a true numerical equality

$$kx_0 + b = 0.$$

Adding the number $(-b)$ to both sides of this numerical equality, we, once again, get a true numerical equality

$$kx_0 = -b.$$

Dividing this equality by k, we obtain

$$x_0 = -\frac{b}{k}.$$

We have proved that if x_0 is the root of Equation (2), then it is necessarily equal to the number $\left(-\frac{b}{k}\right)$.

We now have to check that the number $\left(-\frac{b}{k}\right)$ is really the root of Equation (2):

$$k \cdot \left(-\frac{b}{k}\right) + b = 0.$$

We see that it is actually so.

Hence, we have proved that Equation (2) has the only root

$$x_0 = -\frac{b}{k}.$$

SECTION TEST

QUESTIONS

1. What is a first-degree equation in one unknown?
2. What is the constant term of a first-degree equation in one unknown?
3. What is the coefficient of the unknown in a first-degree equation in one unknown?
4. What is the root of an equation in one unknown?
5. What does it mean to solve an equation?
6. How many roots has a first-degree equation in one unknown?

EXERCISES

1. Write three equations of the first degree in one unknown.
2. Solve the following equations applying the above scheme:
(a) $3x - 7 = 0$; (b) $5x = 0$; (c) $-8x - 10 = 0$; (d) $4x + 15 = 0$.

Sec. 9.2. Linear Equations in One Unknown

The following equations

$$7x + 9 = 0; \quad 3x - 5 + 2x - 1 = 2;$$
$$4x - 3 = 3x - 4; \quad 0 = 2x - 7 - x - 1;$$
$$6x - 3 + 2x = 3x - 4 - x + 1$$

may serve as examples of linear equations in one unknown.

In general, *a linear equation in one unknown* x is defined as an equation whose both sides are polynomials of not higher than the first degree with respect to x or zero.

The reader should recall that a polynomial of not higher than the first degree is understood as a first-degree polynomial or a number different from zero. The terms of polynomials standing in both sides of an equation are called the terms of the equation.

The equation $kx + b = 0$, where k and b are any given numbers is a linear equation both for $k \neq 0$ and for $k = 0$. For k different from zero it is, as it was noted above, also termed an equation of the first degree.

Thus, a first-degree equation in one unknown x is a particular case of a linear equation in one unknown x.

The next section will show how linear equations are solved, and in the present section we are going to consider the notion of equivalence of equations.

Two equations are called *equivalent* if any root of the first equation is a root of the second, and any root of the second equation is a root of the first.

Note the following three statements.

(1) *If both sides of a linear equation are multiplied or divided by a number different from zero, then we get an equation equivalent to the original one.*

For instance, multiplying both sides of the equation

$$x - 1 = 2x - \frac{1}{7} \tag{1}$$

by 7, we get the equation

$$7x - 7 = 14x - 1, \tag{2}$$

which is equivalent to the original equation since, if the number x_0 is the root of Equation (1), then the following numerical equality is fulfilled:

$$x_0 - 1 = 2x_0 - \frac{1}{7}. \tag{3}$$

But then the numerical equality

$$7x_0 - 7 = 14x_0 - 1, \tag{4}$$

is fulfilled showing that x_0 is the root of Equation (2). Conversely, if x_0 is the root of Equation (2), then the numerical equality (4) is fulfilled, but then the numerical equality (3) is also fulfilled thus indicating that x_0 is the root of Equation (1).

Henceforward, we shall say "let us multiply the equation by the number k" instead of "let us multiply the left-hand and right-hand sides of the equation by k".

(2) *If a term of an equation is transposed from one side of an equation to the other with a reverse sign, then we get an equation equivalent to the original one.*

For instance, the equations

$$3x - 2 = 2x + 3 \tag{5}$$

and

$$3x = 2x + 3 + 2 \tag{6}$$

are equivalent.

To obtain Equation (6), we transposed the term (—2) (reversing its sign) in Equation (5) from its left-hand side to the right-hand side.

Formally, the equivalence of Equations (5) and (6) is proved in the following way. Let x_0 be the root of Equation (5). Then the following numerical equality is fulfilled:

$$3x_0 - 2 = 2x_0 + 3, \tag{7}$$

from which there follows the validity of the numerical equality

$$3x_0 = 2x_0 + 3 + 2. \tag{8}$$

Equality (8) shows that x_0 is the root of Equation (6). If now x_0 is the root of Equation (6), then the numerical equality (8) is fulfilled, and, consequently, also the numerical equality (7) which shows that x_0 is the root of Equation (5).

In future, we shall say "let us transpose the term of the given equation from one of its sides to the other", implying that in this process the sign of the term is reversed.

(3) *If in one side of an equation we collect like terms, then we obtain an equation equivalent to the original one.*

For instance, the equations

$$x + x + 2 - 1 = 0$$

and

$$2x + 1 = 0$$

are equivalent. Formally, the proof of this statement is carried out just as in the above examples.

Let us show how it is possible to solve the first-degree equation

$$kx + b = 0 \quad (k \neq 0), \tag{9}$$

using the fact of equivalence of equations.

Transposing the constant term of Equation (9) to the right-hand side, we get the equation

$$kx = -b \tag{10}$$

which is equivalent to Equation (9). Dividing both sides of Equation (10) by the number k, we get the equation

$$x = -\frac{b}{k}, \tag{11}$$

equivalent to Equation (10), and hence, also to Equation (9). Obviously, Equation (11) has the only root $x_0 = -\frac{b}{k}$. Consequently, Equation (9) also has the only root $x_0 = -\frac{b}{k}$.

Thus, in order to solve Equation (9), it is necessary:

(1) to transpose the constant term of this equation to the right, reversing the sign of the number b;

(2) to divide both sides of the obtained equation by the coefficient of the unknown.

Then the number obtained in the right-hand side of the last equation is just the only root of Equation (9).

SECTION TEST

QUESTIONS

1. What is a linear equation in one unknown?
2. What equations are called equivalent?

EXERCISES

1. Give five examples of linear equations in one unknown.
2. Find out whether the indicated pairs of equations are equivalent:
(a) $2x + 3 = 0$ and $2x = -3$;
(b) $3x - 7 = 4x - 3$ and $0 = (4x - 3) - (3x - 7)$;
(c) $-3x - 7 = 0$ and $3x + 7 = 0$;
(d) $-2x + 3 = 0$ and $2x + 3 = 0$;
(e) $3x - 7 + 2x - 3 = x$ and $4x - 10 = 0$?

Sec. 9.3. Solving Linear Equations in One Unknown

Example 1. Solve the equation

$$3x - 5 + 2x - 1 = 0. \tag{1}$$

Collecting like terms in the left-hand side of this equation, we get the equation

$$5x - 6 = 0, \tag{2}$$

which is equivalent to Equation (1). But Equation (2) has only one root

$$x_0 = \frac{6}{5}.$$

Consequently, Equation (1) also has the same unique root: $x_0 = \frac{6}{5}$.

Example 2. Solve the equation

$$3x + 2 - 2x + 5 - x = 0. \tag{3}$$

Collecting like terms in its left-hand side, we get the contradictory equality

$$7 = 0.$$

Hence it follows that Equation (3) is contradictory, it has no roots: there is no such a number x_0 which would satisfy Equation (3). If such a number x_0 existed, then, substituting it for x into Equation (3), no contradiction would be obtained.

Example 3. Solve the equation

$$4x + 7 - x - 6 - 3x - 1 = 0. \tag{4}$$

On collecting like terms, we get

$$0 = 0. \tag{5}$$

Thus, we obtained a true numerical equality: zero equals zero. But we have to explain it from the viewpoint of the set problem.

Since Equality (5) contains no x, it is true for any numerical value of x. But then for any numerical value of x the left-hand side of Equation (4) becomes equal to zero. And this means that any real number is a root of Equation (4).

Thus, we have found all the roots of Equation (4): any real number is a root of this equation.

Example 4. Solve the equation

$$4x + 6 = 2x - 10. \tag{6}$$

Transposing all terms of its right-hand side to the left-hand side and collecting like terms, we get

$$(4x + 6) - (2x - 10) = 0, \tag{7}$$

$$2x + 16 = 0. \tag{8}$$

Equation (8) has the only root

$$x_0 = -8.$$

The obtained number $x_0 = -8$ is not only the root of Equation (8) but also the root of Equation (6) since Equations (6) and (7), the same as Equations (7) and (8), are equivalent.

Example 5. Solve the equation

$$2x + 8 = 2x + 6. \tag{9}$$

Transposing the terms of the right-hand side of this equation to the left and collecting like terms, we get a contradictory equality

$$2 = 0.$$

This shows that Equation (9) has no roots, that is, there is no such a number x_0 which would satisfy this equation. If such a number x_0 existed, then, substituting it for x into the equation and subtracting the same number $(2x_0 + 6)$ from both sides, we would not have obtained the above contradiction.

Example 6. Solve the equation

$$2x + 1 = 3x + 1 - x. \tag{10}$$

Transposing the terms from the right-hand side to the left, we get a true numerical equality

$$0 = 0. \tag{11}$$

Equality (11) shows that Equation (10) turns into a true numerical equality for any numerical value of x. That is, any real number is a root of Equation (10). We may also say that Equation (10) is actually an identity.

Remark. We have considered six examples of linear equations in one unknown x. Any such equation can be solved by transposing all the terms of its right-hand side to the left-hand side and collecting like terms. As a result we shall get:

either a first-degree equation which leads to a unique root;

or the equality $0 = 0$ showing that the original equation is really an identity, that is, showing that any real number is a root of the original equation;

or a contradictory equality indicating that the original equation has no roots.

SECTION TEST

QUESTIONS

1. Is it possible for a linear equation in one unknown: (a) to have no roots; (b) to have a unique root; (c) to have infinitely many roots?

EXERCISES

1. Solve the following equations:
(a) $x + 3 = 2x - 4$; (b) $2x - 4 = 7x + 2$;
(c) $x + 4 = x + 2$; (d) $2x - 6 = 3x$;
(e) $3x - 5 = -2x + 7 + 5x - 12$;
(f) $5x = 6x$; (g) $x - 1 + 3x - 5 = (x - 5) - (x - 3) + (x + 1)$;
(h) $2x + 5 - 7x + 2 = 3$; (i) $7 = 5x + 2 - 3x + 1$.

Sec. 9.4. Solving Problems with the Aid of Linear Equations

Consider the solution of the following old problem.

There flew a flock of geese. Upon meeting the flock, a solitary goose said, "Greetings, hundred geese!" To which the leader of the flock, the oldest goose, replied, "No, you are mistaken. We are not a hundred in number. There are fewer of us than you may think. If the number of us were to be doubled, and then half of it added, and then quarter of it added, plus you, then we would equal a hundred, but now... Just be quick to grasp how many geese are in front of you."

Solution. We have to find out how many geese there were in the flock. Let us denote this number by x. The leader said that if the number of the geese were

(a) to be doubled (or multiplied by 2), that is, plus x;

(b) half of this number added, that is, plus $\frac{1}{2}x$;

(c) quarter of this number added, that is, plus $\frac{1}{4}x$;

(d) and plus one more goose, that is, plus 1, that is, the old goose said that if the number of geese were

$$x+x+\frac{1}{2}x+\frac{1}{4}x+1,$$

then they would equal 100.

Hence,

$$x+x+\frac{1}{2}x+\frac{1}{4}x+1=100.$$

Thus, we have obtained a linear equation in one unknown. Solving this equation, we find its only root $x_0 = 36$, and this means that there were 36 geese in the flock

Here is another problem.

Father is 50 years old, and his son is 20. How many years ago was father three times older than his son?

Solution. Let us denote the desired number of years by x, then x years ago father was $(50 - x)$ years old, while his son was $(20 - x)$. Since at that time father was three times older than his son, we have

$$50-x=3(20-x).$$

And so, we have obtained a linear equation in one unknown. Solving this equation, we find its only root $x_0 = 5$. Hence, five years ago father was three times older than his son.

SECTION TEST

EXERCISES

1. It is required to change one rouble to 2 and 5-copeck coins with the total number of coins equal to 26. How many 2-copeck coins must be obtained?

2. It takes a steamer 3 hours to cover its path when going with the stream and 5 hours to return back, that is, to go against the stream. The rate of flow of the river is 5 km/h. What is the speed of the steamer in stagnant water?

Chapter 10

THE FUNCTION $y = x^2$

Sec. 10.1. Cartesian System of Coordinates in a Plane

Let us be given in a plane two mutually perpendicular axes of coordinates (the x-axis and the y-axis) with the point of intersection O which is the origin of coordinates, that is, the initial point of either of these axes.

Thus, the coordinate system Oxy is said to have been defined in a plane. It is called a *rectangular coordinate system* or *Cartesian system*

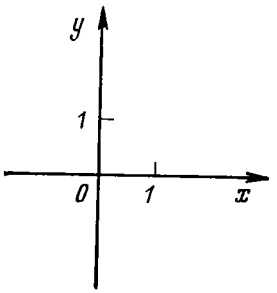


Fig. 3

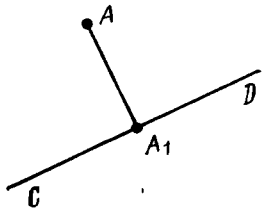


Fig. 4

of coordinates after the French mathematician and philosopher Descartes (which in Latin is Cartesius) who introduced this important notion in mathematics.

The x- and y-axes may have different units of length, or scale units (we shall consider axes with equal scale units if not otherwise stipulated).

The x-axis is also called the *axis of abscissas*, and the y-axis the *axis of ordinates*. The point O of intersection of the coordinate axes is called the *origin of coordinates*. A plane in which a Cartesian system of coordinates is given is called the *coordinate plane*.

On paper, the axis of abscissas (x-axis) is ordinarily represented as a horizontal straight line directed rightwards, and the axis of ordinates (y-axis) as a vertical line directed upwards (Fig. 3).

So, we agree to follow this tradition unless some reasons will make us divert from this convention. Sometimes, we shall also find it convenient to replace the letters x and y by other letters z, t, s, u, ...

The *projection* of a point on some straight line is defined as the foot of the perpendicular dropped from it on this line. Figure 4

represents the straight line CD, point A and point A_1, which is the projection of A on CD.

Let there be given in a plane a rectangular coordinate system Oxy. Let us take a point A in this plane (Fig. 5).

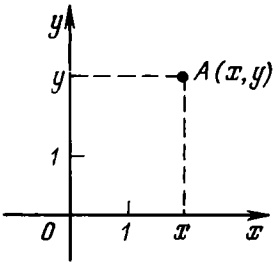


Fig. 5

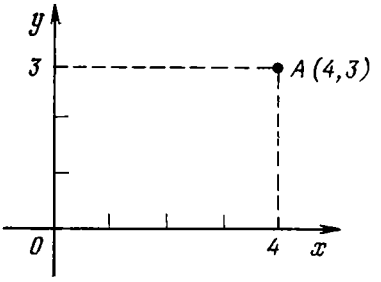


Fig. 6

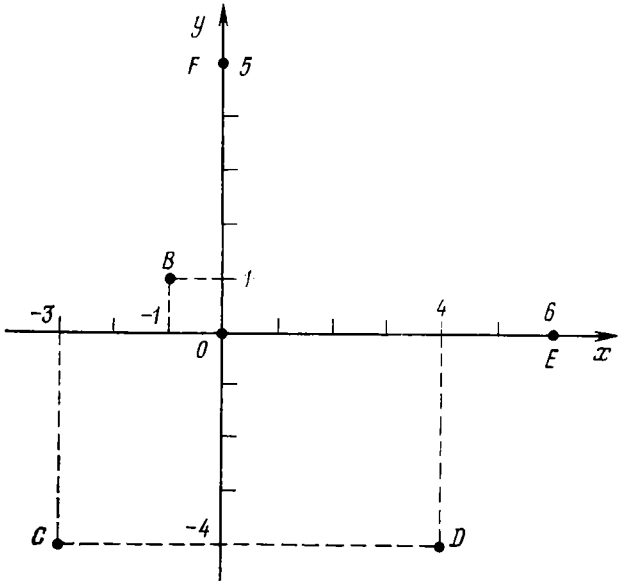


Fig. 7

The *abscissa* of the point A is defined as the coordinate x of its projection on the axis of abscissas (x-axis).

The *ordinate* of the point A is defined as the coordinate y of its projection on the axis of ordinates (y-axis).

The abscissa x and ordinate y of the point A are called the *coordinates* of this point.

The coordinates of a point are usually written in parentheses to the right of the letter denoting this point: $A\ (x, y)$, the first letter denoting the abscissa, the second the ordinate. For example, the point A represented in Fig. 6 has the abscissa $x = 4$ and ordinate $y = 3$, therefore we write $A\ (4, 3)$. Represented in Fig. 7 are the rectangular coordinate system Oxy and the following points:

$O\ (0, 0)$; $B\ (-1, 1)$; $C\ (-3, -4)$; $D\ (4, -4)$; $E\ (6, 0)$; and $F\ (0, 5)$.

It is important to note that if a rectangular coordinate system is given in a plane, then every point in the plane is associated with a pair of numbers (x, y), that is, with the coordinates of this point, and, at the same time, an arbitrary pair of numbers (x, y) may be regarded as the coordinates of a certain point in the plane.

It should be borne in mind that if a pair consists of different numbers, then, interchanging these numbers, we get another pair of numbers defining another point in the plane.

That is why the coordinates (x, y) of the point A are often called an *ordered pair of numbers*. The abscissa x of the point A is also called the first coordinate, and the ordinate y the second coordinate of the point A.

Thus, if a rectangular coordinate system Oxy is given in a plane, then

(1) every point of the plane is associated with an ordered pair of numbers (coordinates of the point);

(2) different points of the plane are associated with different pairs of numbers;

(3) every ordered pair of numbers corresponds to some point (strictly to one, by (2)) of the plane.

These three statements may be replaced by the following words:

There is one-to-one correspondence between points of a plane and ordered pairs of numbers.

SECTION TEST

QUESTIONS

1. What is a Cartesian coordinate system in the plane?
2. How are the coordinates of a point defined?
3. How should we understand the following statement: the set of all points of a coordinate plane is in one-to-one correspondence with the set of all ordered pairs of numbers?

EXERCISES

1. Mark the following points in the coordinate plane:
$A_1\ (-5, 6)$; $A_2\ (-3, -2)$; $A_3\ (2, -1)$; $A_4\ (1, 3)$;
$A_5\ (0, -3)$; $A_6\ (-2, 0)$; $A_7\ (0, 0)$.

Sec. 10.2. Quadrants. Coordinates of Symmetric Points

Looking at the Cartesian coordinate system we see that it divides the plane into four parts called *quadrants* (or coordinate angles or coordinate quarters). Quadrants are usually denoted by the Roman numerals, I, II, III, IV (Fig. 8).

If we exclude the points lying on the coordinate axes, then we may say that:

points of Quadrant I have coordinates (x, y) such that $x > 0$, $y > 0$;

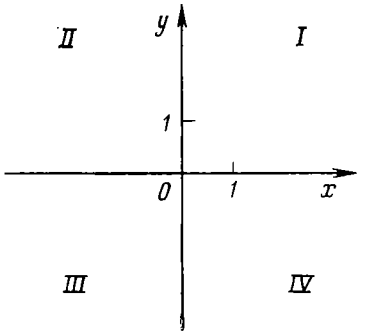


Fig. 8

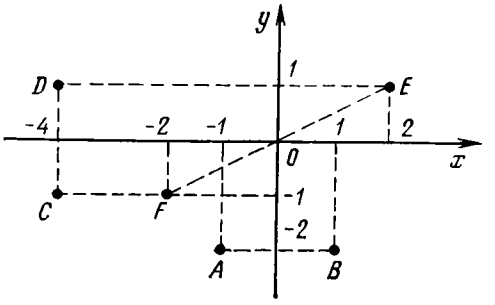


Fig. 9

points of Quadrant II have coordinates (x, y) such that $x < 0$, $y > 0$;

points of Quadrant III have coordinates (x, y) such that $x < 0$, $y < 0$;

points of Quadrant IV have coordinates (x, y) such that $x > 0$, $y < 0$.

For instance, the point B $(-1, 1)$, in Fig. 7 belongs to Quadrant II; the point D $(4, -4)$ belongs to Quadrant IV.

It is easy to see that:

the abscissa of a point is equal to zero if and only if this point lies on the y-axis;

the ordinate of a point is equal to zero if and only if this point lies on the x-axis.

For instance, in Fig. 7 the point F lies on the y-axis and has abscissa $x = 0$, the point E lies on the x-axis and its ordinate $y = 0$.

Note also that the origin, point O, and only this point has both coordinates equal to zero.

Two points A_1 (x_1, y_1) and A_2 (x_2, y_2) are called:

symmetric with respect to the axis of ordinates (y-axis) if their coordinates satisfy the equalities

$$x_1 = -x_2 \quad \text{and} \quad y_1 = y_2;$$

symmetric with respect to the axis of abscissas (x-axis) if their coordinates satisfy the equalities

$$x_1 = x_2 \quad \text{and} \quad y_1 = -y_2;$$

symmetric with respect to the origin (point O) if their coordinates satisfy the equalities

$$x_1 = -x_2 \quad \text{and} \quad y_1 = -y_2.$$

For instance, in Fig. 9 the points A and B are symmetric with respect to the axis of ordinates, points C and D are symmetric with respect to the axis of abscissas, and points E and F are symmetric with respect to the origin.

SECTION TEST

QUESTIONS

1. For what points is abscissa equal to zero?
2. For what points is ordinate equal to zero?
3. For what points is abscissa positive?
4. For what points is ordinate positive?
5. What points are symmetric with respect to the axis of abscissas?
6. What points are symmetric with respect to the axis of ordinates?
7. What points are symmetric with respect to the origin?

EXERCISES

1. For the point A find the points symmetric to it with respect to the x-axis, with respect to the y-axis, and with respect to the origin of coordinates if:
(a) A (1, 2); (b) A (−1, 3); (c) A (2, −3); (d) A (−4, −3) and mark these points on the coordinate plane.

Sec. 10.3. The Notion of Function

Let us begin with a couple of examples.

Example 1. It is known from geometry that the volume of a cube is numerically equal to the cubed length of its edge. This statement is of general character, it refers to any cube. It is only natural to write the equality corresponding to this statement also in general form. Let a denote the length of the edge of the cube, V its volume. Then the mentioned geometrical property can be described as follows:

$$V = a^3 \qquad (a > 0). \tag{1}$$

The inequality in parentheses indicates that we are interested in the above property only for positive values of a since the length of the edge of the cube is a positive number.

Equality (1) is used as a formula by which the volume of any concrete cube is computed. We see that, by virtue of the rule (law) expressed by Formula (1), to each value of the length of the edge a

there corresponds a definite value of the volume V. In such a case we say that V is a *function of* a defined for positive values of a. We also say that V is a function of a defined on the set of positive numbers a.

Example 2. It is known from physics that if a body moves in a straight line with constant velocity, say, 80 (km/h), then the path s (km) covered by this body during time t (h) is determined by the formula

$$s = 80\,t \qquad (t \geqslant 0). \tag{2}$$

By virtue of the rule expressed by Formula (2), to each nonnegative value of t there corresponds a definite value of s. Therefore, in this case we also say that s is a *function* of t defined for nonnegative values of t or defined on the set of nonnegative numbers t.

Sets of numbers are frequently denoted by letters. We could, for instance, denote the set of positive numbers a by A; the set of nonnegative numbers $t \geqslant 0$ by B.

Below, we give the general definition of function.

Let M be a set of numbers, and let to each number x from M a number y be assigned by a certain rule, then y is said *to be a function of* x defined on the set M.

We call x (representing the number to which we assign another one) the *independent variable* or the *argument* and y (which represents the number assigned to the first one) the *dependent variable* or the *function of* x.

The following relation may serve as one more example of function:

$$y = 3x.$$

In this example, the rule of dependence of the variable y on the independent variable x consists in that to each number x the number y equal to three x is assigned. The function expressing this dependence is said to be given by the formula $y = 3x$.

Here are some more examples of functions given by formulas:

$$y = -2x; \qquad y = x; \qquad y = x + 3;$$
$$y = 3x - 4; \qquad y = x^2, \qquad y = 3x^3.$$

The indicated functions are given (defined) for any values of x, that is, for any real numbers x or they are said to be defined on the set of all real numbers.

The letters x, y are often replaced by other letters, especially in concrete cases.

For instance, the area S of a square is the function

$$S = x^2 \qquad (x > 0)$$

of the length of its side x.

The volume V of a cube is the function

$$V = a^3 \qquad (a > 0)$$

of the length of its edge a.

The path s (km) covered during time t (h) by a train moving uniformly with a speed of 80 (km/h) is a function

$$s = 80t \qquad (t \geqslant 0)$$

of t.

Let two points A and B move along the x-axis in the positive direction, the point B being all the time ahead of the point A at a distance of 5 (cm). Then the abscissa of the point B denoted by z is a function of the abscissa x of the point A defined by the formula

$$z = x + 5.$$

A function may be represented in the form of a table. For instance, if air temperature is measured hourly, then to each instant of time $t = 0; 1; 2; \ldots; 24$ there will correspond a definite number T. This correspondence can be written in the form of a table

t	0	1	2	3		14		24
T	16 T_0	16 T_1	15 T_2	15 T_3		25 T_{14}		17 T_{24}

Thus, we have obtained that T is a function of t defined on the set of whole numbers from 0 to 24 and represented by a table. The rule by which to each t from this set there corresponds T is given here not by a formula, but by a table.

Henceforth, we shall study basically functions defined by formulas.

SECTION TEST

QUESTIONS

1. What is function?
2. What is argument?

EXERCISES

1. Give five examples of functions specified by formulas.
2. Represent the perimeter P of a square as a function of the length of its side a.
3. Give an example of a function represented by a table.
4. Compile a table of values of the function $y = 2x^2$ for the values $x = 0; 0.1; 0.2; \ldots; 0.9; 1$.

Sec. 10.4. The Graph of the Function $y = x$

Let there be given in a plane a rectilinear coordinate system Oxy. Let us also specify a straight line serving as the angle bisector of Quadrants I and III, and let then $A\ (x, y)$ be an arbitrary point of this line.

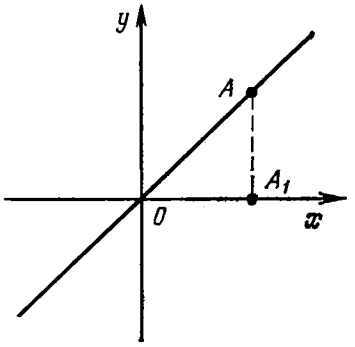


Fig. 10

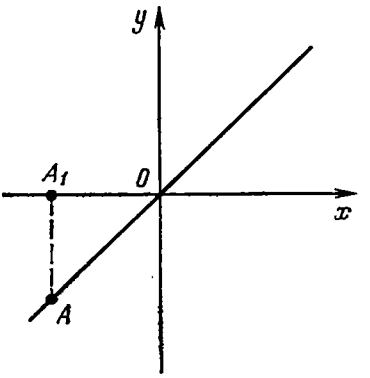


Fig. 11

Marked in Fig. 10 is a point A having a positive abscissa x. Let A_1 be its projection on the axis of abscissas. It is obvious that

$$|\,OA_1\,| = x, \quad |\,A_1A\,| = y.$$

Marked in Fig. 11 is a point A having a negative abscissa x. As before, A_1 is a projection of the point A on the axis of abscissas, but this time

$$|\,OA_1\,| = -x, \quad |\,A_1A\,| = -y.$$

In either of these cases OA_1A is a right triangle, and its acute angle A_1OA is equal to 45°. But then OA_1A is an isosceles triangle and $|\,OA_1\,| = |\,A_1A\,|$, whence we obtain that

$$y = x. \tag{1}$$

We have obtained Equality (1) expressing the relation between the abscissa x and ordinate y of an arbitrary point A of the given straight line. However, when deriving this equality we excluded the case when the point A coincides with the origin O. But it is directly seen that in this case Equality (1) is also fulfilled, since in this case $x = 0$ and $y = 0$.

Thus, any point $A\ (x, y)$ of the angle bisector under consideration has the coordinates satisfying Equality (1). But the converse is also true: if a point $A\ (x, y)$ is such that

$$y = x,$$

then it lies on the angle bisector of Quadrants I and III. Indeed, let us first consider a point $A\ (x, y)$ such that $x > 0$. Then $y > 0$. Let A_1 be the projection of the point A on the axis of abscissas. It is obvious that $|OA_1| = |AA_1|$. Hence, the right triangle OA_1A is isosceles, consequently, either of its acute angles is equal to 45°. But then the point $A\ (x, y)$ lies on the angle bisector of Quadrant I. And if $x < 0$, then $y < 0$, and, referring to Fig. 11, we obtain that the point $A\ (x, y)$ lies on the angle bisector of Quadrant III.

Thus, we have shown that:

if a point $A\ (x, y)$ lies on the angle bisector of Quadrants I and III, then $y = x$;

and if a point $A\ (x, y)$ is such that $y = x$, then it lies on the angle bisector of Quadrants I and III.

The angle bisector of Quadrants I *and* III *is said to be the graph of the function* $y = x$.

We also say that the *angle bisector of Quadrants* I *and* III *has the equation* $y = x$.

Finally, we say that the *angle bisector of Quadrants* I *and* III *is the set of points* $A\ (x, y)$ *in the plane whose coordinates satisfy the equation* $y = x$.

SECTION TEST

QUESTIONS

1. What is the graph of the function $y = x$?
2. What equation has the angle bisector of Quadrants I and III?

EXERCISES

1. Prove that the angle bisector of Quadrants II and IV is the graph of the function $y = -x$.

Sec. 10.5. Tables of the Function $y = x^2$

The formula

$$y = x^2 \tag{1}$$

defines the function for all real values of x: to every number x there corresponds (with the aid of this formula) a value of y equal to the square of x.

The function $y = x^2$ is said to be defined on the set of all real values of x. We also say that x is an independent variable running over the indicated set, and y is a function of x.

For particular values of x it is possible to compute the corresponding values of y.

Given below is a table of values of x running over nonnegative integers:

$$x = 0;\ 1;\ 2;\ 3;\ \ldots;\ 11,$$

and the corresponding values of y.

Table 1

x	0	1	2	3	4	5	6	7	8	9	10	11
y	0	1	4	9	16	25	36	49	64	81	100	121

It is seen from this table that $y = 0$ for $x = 0$, that is, if the argument $x = 0$, then the corresponding to it value of the function $y = 0$.

It is also seen that if the independent variable x increases when running through natural numbers, then the function y increases accordingly running through the numbers 1, 4, 9, 16 ... . It is important to note that if x increases without bound, that is, tends to $+\infty$, then its corresponding value of y also tends to $+\infty$.

Thus,

$$y \to +\infty \quad \text{as} \quad x \to +\infty.$$

Here is another table with the values of x given with a step equal to 0.1 over the interval [0, 2] and the corresponding values of y computed.

Table 2 deals with positive values of x and $x = 0$. The same as from Table 1, we see from this table that to a greater value of x there corresponds a greater value of y, that is, if $x_1 < x_2$ and $y_1 = x_1^2$, $y_2 = x_2^2$, then

$$y_1 < y_2.$$

Therefore, the function $y = x^2$ is said to *be increasing* for the values of x indicated in Table 2.

Table 2

x	y	x	y	x	y
0	0	0.7	0.49	1.4	1.96
0.1	0.01	0.8	0.64	1.5	2.25
0.2	0.04	0.9	0.81	1.6	2.56
0.3	0.09	1.0	1.00	1.7	2.89
0.4	0.16	1.1	1.21	1.8	3.24
0.5	0.25	1.2	1.44	1.9	3.61
0.6	0.36	1.3	1.69	2.0	4.00

SECTION TEST

QUESTIONS

1. On what set is the function $y = x^2$ defined?

EXERCISES

1. Compile a table of the values of the function $y = x^2$ with x changing in the interval [2, 3] with a step of 0.1.
2. Compile a table of the values of the function $y = x^2$ with x changing in the interval [2.8, 3] with a step of 0.01.

Sec. 10.6. Continuity of the Function $y = x^2$

Let us consider the first part of Table 2 from the preceding section where the variations of x are given for the interval $[0, 1]$. The following important fact may be revealed from this table: if two values of x differ but little, then the corresponding values of y differ by a small quantity.

For instance, to the values

$$x_1 = 0.1, \quad x_2 = 0.2$$

there correspond the values

$$y_1 = (0.1)^2 = 0.01, \quad y_2 = (0.2)^2 = 0.04,$$

and to the difference

$$x_2 - x_1 = 0.2 - 0.1 = 0.1$$

there corresponds the difference

$$y_2 - y_1 = 0.04 - 0.01 = 0.03.$$

And if

$$x_1 = 0.8, \quad x_2 = 0.9,$$

then

$$y_1 = 0.64, \quad y_2 = 0.81.$$

Whence

$$x_2 - x_1 = 0.1 \quad \text{and} \quad y_2 - y_1 = 0.17.$$

For the second part of Table 2, where x varies on the interval $[1, 2]$ the indicated property that to a small change in x there corresponds a small change in y is not so apparent. For instance, if

$$x_1 = 1.8, \quad x_2 = 1.9,$$

then

$$x_2 - x_1 = 0.1,$$

and

$$y_2 - y_1 = 3.61 - 3.24 = 0.37.$$

But if we compiled a more detailed table with x changing with a step of 0.01 or 0.001 or 0.0001, then we would make sure that the property we are interested in, namely, to a small change in x there corresponds a small change in y, is really evident. To obtain a desired smallness of the change in y, we have to take a table with sufficiently small changes in x.

Given below is Table 3 with x changing on the interval [1, 2] with a step of 0.01.

Table 3

x	y	x	y	x	y	x	y
1.00	1.00	1.06	1.1236	1.91	3.6481	1.96	3.8416
1.01	1.0201	1.07	1.1449	1.92	3.6864	1.97	3.8809
1.02	1.0404	1.08	1.1664	1.93	3.7249	1.98	3.9204
1.03	1.0609	1.09	1.1881	1.94	3.7636	1.99	3.9601
1.04	1.0816	1.10	1.2100	1.95	3.8025	2.00	4.0000
1.05	1.1025	1.90	3.610				

Now the greatest difference between two successive values of y taking place for

$$x_1 = 1.99, \quad x_2 = 2,$$

is equal to

$$y_2 - y_1 = 2^2 - (1.99)^2 = 4 - 3.9601 = 0.0399 < 0.04.$$

So if we want to get a difference between two neighbouring values of y on the interval [1, 2] not exceeding 0.04, then it is necessary to use Table 3 since Table 2 is too rough for this purpose.

Consider one more table, where the values of x approach the number 2 closer and closer. In such cases we say that the variable x tends to the number 2.

Table 4

x	2.1	2.01	2.001	2.0001	2.00001
y	4.41	4.0401	4.004001	4.00040001	4.0000400001

We see that when the variable x tends to the number 2, the variable y tends to the number $4 = 2^2$. This is written as follows:

$$\text{if } x \to 2, \text{ then } y \to 4,$$

or

$$\text{if } x - 2 \to 0, \text{ then } y - 4 \to 0.$$

We may also write:

$$y \to 4 \text{ as } x \to 2,$$

or

$$y - 4 \to 0 \text{ as } x - 2 \to 0.$$

The property which was mentioned above is the property of continuity of the function $y = x^2$ for a given value of the argument x_0.

In the considered example $x_0 = 2$.

The function $y = x^2$ is said to be *continuous* for $x = x_0$ since it possesses the following property:

$$\text{if } x \to x_0, \text{ then } x^2 \to x_0^2.$$

In the next section, we formulate and justify some general properties of the function

$$y = x^2$$

which we succeeded to reveal while considering the tables of particular values of this function.

SECTION TEST

QUESTIONS

1. What is meant by the statement that the function $y = x^2$ is continuous for the value $x = x_0$?

EXERCISES

1. Compile a table of the values of the function $y = x^2$ for

$$x = 2.9; \quad 2.99; \quad 2.999; \quad 2.9999.$$

To what does the variable x tend if we regard this process to be endless? What does $y = x^2$ tend to accordingly?

Sec. 10.7. Properties of the Function $y = x^2$

Let us now formulate and justify some properties of the function

$$y = x^2.$$

(1) If $x = 0$, then $y = 0$.

(2) If $x > 0$, then $y > 0$.

Indeed, multiplying the inequality $x > 0$ by a positive number x, we get $x^2 > 0$, that is, $y > 0$.

(3) For nonnegative values of x the function $y = x^2$ increases, that is, to a larger value of x there corresponds a larger value of y.

In other words, if x_1 and x_2 are nonnegative numbers, $y_1 = x_1^2$, $y_2 = x_2^2$, and $x_1 < x_2$, then

$$y_1 < y_2.$$

Indeed, for $x_1 = 0$ this is Property (2). Let now $0 < x_1$ and $x_1 < x_2$. Then, multiplying the inequality $x_1 < x_2$ by a positive number x_1, we get

$$x_1^2 < x_1 x_2$$

and, multiplying the inequality $x_1 < x_2$ by the positive number x_2, we get

$$x_1 x_2 < x_2^2.$$

Consequently,

$$x_1^2 < x_2^2, \text{ i.e. } y_1 < y_2.$$

(4) If a positive x, increasing without bound, tends to plus infinity, then $y = x^2$ also tends to plus infinity, that is,

$$y \to +\infty \text{ as } x \to +\infty.$$

Indeed, let x tend to plus infinity, running through the natural numbers

$$n = 1;\ 2;\ 3;\ 4;\ \ldots.$$

Then $y = x^2$ traverses the numbers

$$n^2 = 1;\ 4;\ 9;\ 16;\ 25;\ \ldots$$

and obviously, also tends to plus infinity.

Obviously, for intermediate nonintegral values of x this property also holds true since the function $y = x^2$ increases for $x > 0$.

(5) If the sign of x is reversed, the corresponding value of $y = x^2$ remains unchanged.

Indeed,

$$(-x)^2 = x^2.$$

A function possessing these properties is called an *even function*. Thus, the function $y = x^2$ is an even function.

(6) The function $y = x^2$ is continuous, that is, if the variable x tends to a given number x_0, then the value of the function $y = x^2$, corresponding to x, tends to its value corresponding to the number x_0, that is, to the number x_0^2.

This property is written as follows:

$$y \to y_0, \text{ as } x \to x_0,$$

or

$$y - y_0 \to 0 \text{ as } x - x_0 \to 0.$$

Indeed, from the equalities

$$y = x^2, \quad y_0 = x_0^2$$

it follows that

$$y - y_0 = x^2 - x_0^2 = (x - x_0)(x + x_0)$$
$$= (x - x_0)[x - x_0 + 2x_0] = (x - x_0)^2 + 2x_0(x - x_0). \quad (1)$$

If now $x - x_0$ tends to zero, that is $x - x_0 \to 0$, then

$$(x - x_0)^2 \to 0 \quad \text{and} \quad 2x_0(x - x_0) \to 0.$$

Consequently,

$$(x - x_0)^2 + 2x_0(x - x_0) \to 0,$$

that is, we have proved that

$$\text{if } x - x_0 \to 0, \quad \text{then} \quad y - y_0 \to 0.$$

SECTION TEST

QUESTIONS

1. What does it mean that the function $y = x^2$ increases for nonnegative values of x?
2. What does it mean that the function $y = x^2$ is even?
3. What does it mean that the function $y = x^2$ is continuous?

EXERCISES

1. Prove that for negative values of x the function $y = x^2$ decreases, that is, to a larger value of x there corresponds a smaller value of y.
2. Prove that $y \to +\infty$ as $x \to -\infty$.
3. Formulate the properties of the function $y = x^2$.

Sec. 10.8. The Graph of the Function $y = x^2$

To represent the graph of the function

$$y = x^2, \quad (1)$$

we must compute for every real number x the corresponding value of y by Formula (1) and mark (plot) the points (x, y) thus obtained in a plane with a Cartesian coordinate system introduced. The totality of all these points just forms the graph of the function $y = x^2$.

But it is impossible to complete this work since there is an infinitude of indicated points. Still, we can depict the graph of the function $y = x^2$ approximately.

Let us be given separate positive values of x and let us compute the corresponding values of y using Formula (1). We then compile a table from these numbers. Given below is such a table for the

values of x which differ from one another by 0.1 within the interval [0, 3]:

x	y	x	y	x	y	x	y	x	y
0	0	0.7	0.49	1.3	1.69	1.9	3.61	2.5	6.25
0.1	0.01	0.8	0.64	1.4	1.96	2.0	4.00	2.6	6.76
0.2	0.04	0.9	0.81	1.5	2.25	2.1	4.41	2.7	7.29
0.3	0.09	1.0	1.00	1.6	2.56	2.2	4.84	2.8	7.84
0.4	0.16	1.1	1.21	1.7	2.89	2.3	5.29	2.9	8.51
0.5	0.25	1.2	1.44	1.8	3.24	2.4	5.76	3.0	9.0
0.6	0.36								

Let us plot the points (x, y) contained in the table in a plane with a rectangular coordinate system Oxy specified (Fig. 12). We then joint these points with a smooth continuous curve such that

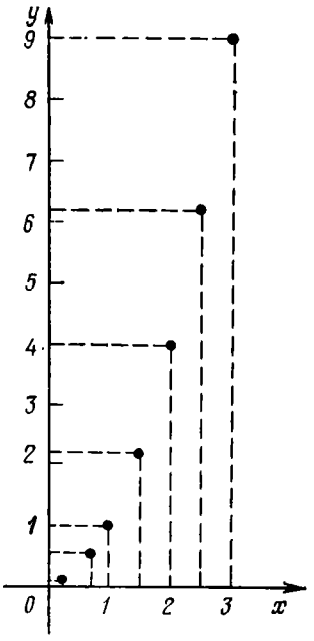


Fig. 12

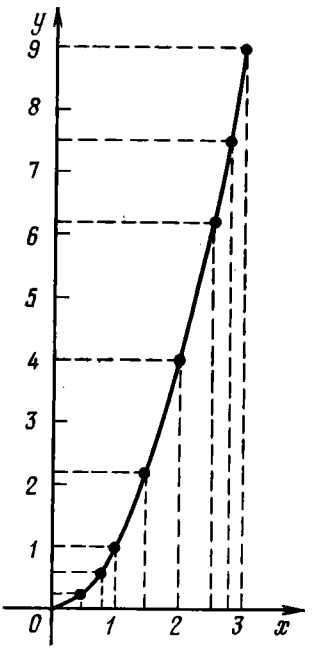


Fig. 13

the ordinate y of its moving point increases together with the abscissa x (Fig. 13). The continuous curve thus obtained may be regarded as an approximate graph of the function $y = x^2$ on the interval [0, 3] of variation of x.

Note that the graph represented in Fig. 13 reflects Properties (1), (2), (3), (6) of the function $y = x^2$ formulated in the preceding section. However, these words need explanations as far as Property (6) is concerned.

A continuous graph can be drawn with one stroke of the pencil, that is, with one continuous motion of the pencil without tearing it off the paper. Thus, if A is a point of a continuous graph having the abscissa x_0 and ordinate y_0, and B is another point of the graph with abscissa x and ordinate y, then, as x tends to x_0, the point B, while moving along the graph, tends, that is, approaches arbitrarily close, to the point A. Consequently, the ordinate y of the point B tends to the ordinate y_0 of the point A. But this is just Property (6) of the function $y = x^2$. By virtue of this property, $x^2 \to x_0^2$ as $x \to x_0$.

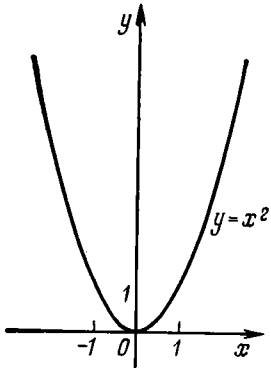


Fig. 14

We see that Property (6) indicates that the graph of the function $y = x^2$ must represent a continuous curve. Therefore we just connected the plotted points with a smooth curve.

It is easy to imagine how the graph of the function $y = x^2$ looks like for large x's. If the abscissa of a moving point of this graph tends to $+\infty$, then, by Property (4), its ordinate y also tends to $+\infty$. We should have in view that y tends to $+\infty$ much more rapidly than x. If, for instance, x traverses the numbers

$$1;\ 2;\ 3;\ 4;\ \ldots,$$

then y, accordingly, traverses the squares of these numbers:

$$1;\ 4;\ 9;\ 16;\ 25;\ 36;\ 49;\ \ldots.$$

By virtue of the property

$$(-x)^2 = x^2,$$

the graph of the function $y = x^2$ is symmetric with respect to the y-axis. Therefore for negative x's we have to construct a curve symmetric with respect to the y-axis.

The graph of the function $y = x^2$ has the form represented in Fig. 14. The curve representing the graph of the function $y = x^2$ is called the *parabola*. We shall often say briefly: "the parabola $y = x^2$".

However, we shall learn later that the graphs of functions of the form

$$y = ax^2 + bx + c,$$

where a, b, c are given numbers and $a \neq 0$, are also called parabolas.

The point of intersection of the parabola with its axis of symmetry is called the *vertex* of the parabola. In the present case this is the point O (0, 0).

Considering the parabola $y = x^2$, we can derive a number of properties of the function $y = x^2$.

Indeed, the parabola $y = x^2$ passes through the origin. This is Property (1) of the function $y = x^2$.

All points of the parabola, except its vertex, are found above the x-axis. For points with abscissa $x > 0$ this is Property (2).

If a point A (x, y) of the parabola moves in it so that its abscissa x is positive and increases, then its ordinate also increases. This is Property (3). But we also see from the graph that if the abscissa x of a moving point of the parabola is negative and increases, then its ordinate y decreases, that is, to larger (negative!) values of x there correspond smaller values of y. Of course, this property can be obtained from Properties (1)-(6) (see exercises below).

The parabola $y = x^2$ is a continuous curve (Property (6)) symmetric with respect to the y-axis (Property (5)).

SECTION TEST

QUESTIONS

1. What is the name of the curve which is the graph of the function $y = x^2$?
2. What does the continuity property of this graph consist in?
3. What point is called the vertex of the parabola $y = x^2$?
4. What line is the axis of symmetry of the parabola $y = x^2$?
5. How does the ordinate of a moving point of the parabola change when its abscissa tends to $+\infty$? to $-\infty$?

EXERCISES

1. Show that Properties (2) and (6) of the function $y = x^2$ imply that $y > 0$ for all $x \neq 0$.
2. Show that Properties (3) and (6) of the function $y = x^2$ imply that for nonpositive values of x the function $y = x^2$ decreases that is, if $x_1 < x_2 \leqslant 0$ and $y_1 = x_1^2$, $y_2 = x_2^2$, then $y_1 > y_2$.

Chapter 11

SQUARE ROOTS

In the present chapter we shall consider the following problem: for a given real number b find a real number a such that $a^2 = b$.

Sec. 11.1. The Concept of a Square Root

Let us be given a real number a. Raising it to the second power (or squaring it), we get a real number $b = a^2$ which is called the *square* of the number a. Let us show that b is a nonnegative number. Indeed,

if $a = 0$, then

$$b = 0^2 = 0 \cdot 0 = 0;$$

if $a > 0$, then, multiplying this inequality by a positive number a, we get

$$b = a^2 > 0;$$

and if $a < 0$, then, multiplying this inequality by a negative number a, we get

$$b = a^2 > 0.$$

Thus, we have shown that *for any real number a the following inequality holds true*:

$$a^2 \geqslant 0,$$

that is, *the square of any real number is a nonnegative number.*

From the aforegoing it follows that *there is no such a real number whose square is equal to a negative number.*

Using the graphical method, let us now show that for any nonnegative number b there is a real number a such that

$$a^2 = b.$$

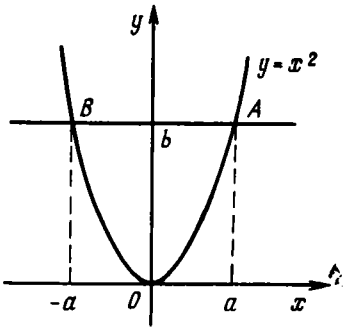


Fig. 15

For $b = 0$ we have to find a number a such that $a^2 = 0$. But then $a = 0$ since $0^2 = 0$, and if $a \neq 0$, then $a^2 > 0$.

Thus, there is a unique number such that its square is equal to the number b, this is the number 0.

Let now $b > 0$. We draw in the rectangular coordinate system Oxy the graph of the parabola (Fig. 15)

$$y = x^2.$$

We lay off on the y-axis from the origin upwards a line segment of length b and through its upper end point draw a straight line parallel to the x-axis. This line intersects the parabola $y = x^2$ at two points: A and B (see Fig. 15).

Let the abscissa of the point A be equal to the number a. Then the abscissa of the point B is the number $(-a)$ since the points A and B are symmetric with respect to the y-axis.

It is obvious that the squares of the numbers a and $(-a)$ are equal to b:

$$a^2 = (-a)^2 = b.$$

There is no other real numbers whose square would be equal to b.

The square root of a given number is defined as such a number whose square is equal to the given number.

It follows from this definition that:

(1) there are only two square roots of any positive number b. They are equal to each other in absolute value, but have different signs, that is, one of the roots is positive and the other is negative;

(2) the square root of zero is unique, it equals zero;

(3) there is no such a real number which would be the square root of a negative number.

Remark 1. Later on, we shall introduce square roots of negative numbers, but they will be not real but so-called complex numbers.

Remark 2. We say that the square root of a negative number does not exist, meaning by this that there is no real number whose square is a negative number.

Example.

17 and (-17) are square roots of 289 since $17^2 = (-17)^2 = 289$.

The numbers $\frac{1}{7}$ and $\left(-\frac{1}{7}\right)$ are the square roots of $\frac{1}{49}$ since $\left(\frac{1}{7}\right)^2 = \left(-\frac{1}{7}\right)^2 = \frac{1}{49}$.

The numbers $\frac{5}{3}$ and $\left(-\frac{5}{3}\right)$ are the square roots of $\frac{25}{9}$ since $\left(\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right)^2 = \frac{25}{9}$.

The number 0 is the only square root of 0.

There are no square roots of the number -4.

SECTION TEST

QUESTIONS

1. Can the square of a real number be a negative number?
2. What is the square root of a given number?
3. How many square roots of a nonnegative number are there?
4. Are there real numbers which are square roots of a given negative number?

EXERCISES

1. Find the square root of each of the indicated numbers:

(a) $\frac{1}{169}$; (b) 625; (c) 1,000,000; (d) 1.44;

(e) 30.25; (f) 0.0256; (g) 0; (h) -9.

Sec. 11.2. The Arithmetic Square Root

An *arithmetic square root* of a nonnegative number is defined as a nonnegative square root of this number.

For instance, the numbers

$$0;\ 1;\ 2;\ 15$$

are arithmetic square roots of the respective numbers

$$0;\ 1;\ 4;\ 225.$$

Thus, *the arithmetic square root of a given nonnegative number b is defined as a nonnegative number a such that $a^2 = b$*, this number a being denoted by the symbol $\sqrt{b}$:

$$a = \sqrt{b}.$$

This notation is read as follows: "a is an arithmetic square root of the number b". Therefore

$$\sqrt{0} = 0;\ \sqrt{1} = 1;\ \sqrt{4} = 2;\ \sqrt{9} = 3;$$
$$\sqrt{16} = 4;\ \sqrt{225} = 15;\ \sqrt{25} = 5;$$
$$\sqrt{\frac{1}{49}} = \frac{1}{7};\ \sqrt{\frac{25}{9}} = \frac{5}{3}.$$

Sometimes, instead of the words "an arithmetic square root of a number", we say "an arithmetic value of the square root of a number".

The symbol $\sqrt{b}$ for $b \geqslant 0$ will always be understood as an arithmetic square root of b. The symbol $\sqrt{b}$ will often be named the square root of b with the adjective "arithmetic" omitted (for the sake of brevity), but implied.

Of the two square roots of the positive number b one is arithmetic, that is, equal to $\sqrt{b}$, the other being equal to $(-\sqrt{b})$.

We recall here that the square root of 0 is only one: $\sqrt{0} = 0$. Hence, *for every nonnegative number b there exists only one arithmetic square root*: $\sqrt{b}$.

In the present chapter we shall study only arithmetic square roots.

Note that if nonnegative numbers b_1 and b_2 are such that $b_1 < b_2$, then $\sqrt{b_1} < \sqrt{b_2}$.

Indeed, let us suppose that $\sqrt{b_1} \geqslant \sqrt{b_2}$, then, squaring this inequality, we get $b_1 \geqslant b_2$ which contradicts to the original inequality $b_1 < b_2$.

Thus, we have proved the following: *let there be given two nonnegative numbers; if the square of the first of them exceeds the square of the second, then the first number is greater than the second.*

The aforegoing also implies that *if two nonnegative numbers are equal to each other, then their arithmetic square roots are equal.* Or in other words: if the squares of two nonnegative numbers are equal, then these numbers are equal to each other.

SECTION TEST

QUESTIONS

1. What is an arithmetic square root?
2. How many arithmetic square roots of a given number are there?
3. Can arithmetic square roots of unequal numbers be equal?

EXERCISES

1. Find the arithmetic square root of each of the given numbers

(a) 3; (b) 0.49; (c) $\frac{1}{9}$; (d) 0; (e) -7.

2. Compute the following:

(a) $\sqrt{\frac{1}{169}}$; (b) $\sqrt{30.25}$; (c) $\sqrt{0.0121}$.

Sec. 11.3. The Square Root of a Natural Number

The square of a natural number is a natural number. But not any natural number is the square of some natural number.

Among natural numbers not exceeding 20, only four:

$$1;\ 4;\ 9;\ 16$$

are squares of natural numbers.

And among natural numbers less than 1000, there are only 31, that is, about 3% all in all. Here they are:

1; 4; 9; 16; 25; 36; 49; 64; 81; 100; 121; 144; 169; 196; 225; 256; 289; 324; 361; 400; 441; 484; 529; 576; 625; 676; 729; 784; 841, 900, 961.

Considering natural numbers not greater than 10,000, we see that among them there are only 100 numbers (that is, 1%) which are squares of natural numbers, namely:

$$1^2;\ 2^2;\ 3^2;\ \ldots;\ 100^2.$$

Taking arithmetic square roots of these numbers, we get the respective numbers:

$$1;\ 2;\ 3;\ \ldots;\ 100.$$

We see that considering large natural numbers, we seldom come across squares of natural numbers.

It is of interest that if a natural number is not a square of some other natural number, then it is a square of an irrational number. Thus, the square roots

$$\sqrt{2};\ \sqrt{3};\ \sqrt{5};\ \sqrt{6};\ \sqrt{7};\ \sqrt{8};\ \sqrt{10};\ \sqrt{11};\ \sqrt{12};\ \ldots$$

are irrational numbers.

Of course, we have to prove this statement. We are going to prove it taking $\sqrt{2}$ as an example.

Example 1. Prove that the number $\sqrt{2}$ is irrational.

Suppose that $\sqrt{2}$ is a rational number, that is,

$$\sqrt{2}=\frac{p}{q},$$

where p and q are natural numbers. We shall regard that the fraction $\frac{p}{q}$ is irreducible; otherwise we would have reduced it beforehand. On squaring the indicated equality and multiplying by q^2, we get

$$2q^2 = p^2.$$

The left-hand member of this equality is a natural number divisible by 2. But then p^2 is also divisible by 2. Then p must also be divisible by 2 (since if p were not divisible by 2, then p^2 would also be not divisible by 2). Consequently, p^2 must be divisible by 4. But then q^2 must be divisible by 2, consequently, q must be divisible by 2. Thus, having supposed that $\sqrt{2}$ is a rational number expressed by an irreducible fraction $\frac{p}{q}$, we obtained that both p and q are divisible by 2, that is, we have arrived at a contradiction. Hence, $\sqrt{2}$ is an irrational number.

The proof of irrationality of the square root of any natural number which is not a square of some natural number may be carried out in the following way.

Let there be given a natural number N which is not a square of some natural number. Suppose $\sqrt{N}$ is a rational number. Let us write it in the form of an irreducible fraction $\frac{p}{q}$ $(q > 1)$. Then

$$\sqrt{N}=\frac{p}{q}.$$

Squaring both sides of this equality, we get

$$N=\frac{p^2}{q^2}.$$

But this equality is impossible since N is a natural number, and the fraction $\frac{p^2}{q^2}$, the same as the fraction $\frac{p}{q}$, is irreducible.

SECTION TEST

QUESTIONS

1. Can the square root of a prime number be a rational number?
2. Can the square root of a natural number be a rational number?

EXERCISES

1. Prove that the following numbers are irrational:
(a) $\sqrt{3}$; (b) $\sqrt{5}$; (c) $\sqrt{7}$; (d) $\sqrt{11}$.
2. Prove that the following numbers are irrational:
(a) $\sqrt{80}$; (b) $\sqrt{972}$; (c) $\sqrt{1152}$.

Sec. 11.4. Approximate Computation of Square Roots

We know that $\sqrt{2}$ is an irrational number. Consequently, its decsmal expansion

$$\sqrt{2}=\alpha_0.\ \alpha_1\alpha_2\alpha_3\ \ldots$$

ii nonterminating and nonrepeating.

Let us show how the number α_0 and the successive digits α_1, α_2, α_3, . . . are computed.

From the inequalities

$$1<2<4,$$

it follows, as is shown in Sec. 11.2, that

$$\sqrt{1}<\sqrt{2}<\sqrt{4},\ \text{i.e.}\ 1<\sqrt{2}<2.$$

Hence,

$$\alpha_0 = 1.$$

Now, we shall seek for α_1. To this end, consider the numbers

$$1.0;\quad 1.1;\quad 1.2;\quad 1.3;\quad \ldots,\quad 1.9;\quad 2.$$

The number $\sqrt{2}$ is found somewhere between two of these numbers. To find out between which of them, we shall square them in turn. We have

$$1.0^2 = 1;\quad 1.1^2 = 1.21;\quad 1.2^2 = 1.44;$$
$$1.3^2 = 1.69;\quad 1.4^2 = 1.96;\quad 1.5^2 = 2.25.$$

There is no need in further computation since we see that

$$1.4^2 < 2 < 1.5^2,$$

or

$$1.4<\sqrt{2}<1.5.$$

This shows that

$$\alpha_1 = 4.$$

To find α_2 we have to consider the squares of the following numbers:

$$1.40^2 = 1.96; \quad 1.41^2 = 1.9881; \quad 1.42^2 = 2.0164.$$

We see that

$$1.41^2 < 2 < 1.42^2,$$

or

$$1.41 < \sqrt{2} < 1.42.$$

Therefore

$$\alpha_2 = 1.$$

Thus, we have found the integral part and the first two digits to the right of the decimal point in the decimal expansion of $\sqrt{2}$. And so, we have found the approximate value of $\sqrt{2}$:

$$\sqrt{2} \approx 1.41$$

with an accuracy to the second decimal place.

Continuing this process, we shall find approximate values of $\sqrt{2}$ with the needed number of digits to the right of the decimal point. In other words, continuing this process we shall compute $\sqrt{2}$ within any required decimal place. This process may be applied to approximate calculation of the arithmetic square root of any natural number which is not a square of a natural number.

But we usually need not carry out such calculations. There are detailed tables of approximate values of square roots of natural numbers. Besides, hand-held calculators have found ever broader application. With the aid of such a computing device, we can find out in no time the approximate value of the square root of a given natural number. Square roots of numbers are also computed with the aid of a slide rule. When studying the logarithmic function, we shall dwell on the slide rule in more detail.

SECTION TEST

QUESTIONS

1. How do we find approximate values of the square roots of natural numbers?

2. What does it mean to compute $\sqrt{N}$ with an accuracy to the seventh decimal place, where N is a prime number?

EXERCISES

1. Compute the following numbers to within the third decimal place: a) $\sqrt{3}$; (b) $\sqrt{5}$; (c) $\sqrt{7}$ and check the results against the table of square roots.

2. Compute the number $\sqrt{2}$ with an accuracy to the 5th decimal digit.

Sec. 11.5. Properties of Arithmetic Square Roots

Let a and b denote any nonnegative numbers. Then the following equalities hold true:

$$(\sqrt{a})^2 = a, \tag{1}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}, \tag{2}$$

and if also $b > 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \tag{3}$$

Finally, for any real number a the following equality holds true:

$$\sqrt{a^2} = |a|. \tag{4}$$

Equality (1) is true by definition: in fact, $\sqrt{a}$ for nonnegative number a is a number whose square is equal to a.

Equality (2) states that the *root of a product of nonnegative numbers is equal to the product of the roots of these numbers.*

Note that in this formulation, for the sake of brevity, we have written simply "root" instead of "arithmetic square root".

Both members of Equality (2) are nonnegative numbers, and their squares are equal to the same number ab:

$$(\sqrt{ab})^2 = ab, \quad (\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab.$$

But then, as we know from Section 11.2, the numbers themselves are also equal to each other.

Equality (3) states that the *root of a quotient from dividing a nonnegative number by a positive number is equal to the quotient of the roots of these numbers.*

Equality (3) is proved just in the same way as Equality (2). We square both of its sides:

$$\left(\sqrt{\frac{a}{b}}\right)^2 = \frac{a}{b}, \quad \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

Since the squares are equal, the left-hand side of Equality (3) is equal to the right-hand side.

The square of any real number is nonnegative. Therefore, the left-hand member of Equality (4) really expresses an arithmetic root of a nonnegative number.

By property (1), $(\sqrt{a^2})^2 = a^2$. Let us now show that $|a|^2 = a^2$. Indeed,

if $a = 0$, then $|a| = 0$, and therefore $|a|^2 = 0 = a^2$;

if $a > 0$, then $|a| = a$, and therefore $|a|^2 = a^2$;

if $a < 0$, then $|a| = -a$, and therefore $|a|^2 = (-a)^2 = a^2$.

Thus, we have shown that the square of the left-hand member of Equality (4) is equal to the square of its right-hand member, hence, the members themselves are also equal to each other.

Equalities (1), (2), (3) and (4) help us simplify numerical expressions containing square roots.

Example 1.

$$\sqrt{80}=\sqrt{16\cdot 5}=\sqrt{16}\cdot\sqrt{5}=4\sqrt{5}.$$

Example 2.

$$\sqrt{\frac{27}{25}}=\sqrt{\frac{9\cdot 3}{25}}=\sqrt{\frac{9}{25}}\sqrt{\frac{3}{1}}=\frac{3}{5}\cdot\sqrt{3}.$$

Example 3.

$$(\sqrt{11}-\sqrt{7})(\sqrt{11}+\sqrt{7})=(\sqrt{11})^2-(\sqrt{7})^2=11-7=4.$$

Example 4.

$$\sqrt{8}\cdot\sqrt{32}=\sqrt{8\cdot 32}=\sqrt{16^2}=16.$$

Example 5.

$$\begin{aligned}&(2\sqrt{8}+3\sqrt{5}-7\sqrt{2})(\sqrt{72}+\sqrt{20}-4\sqrt{2})\\&\quad=(2\sqrt{4}\sqrt{2}+3\sqrt{5}-7\sqrt{2})(\sqrt{2}\sqrt{36}+\sqrt{4}\sqrt{5}-4\sqrt{2})\\&\quad=(4\sqrt{2}+3\sqrt{5}-7\sqrt{2})(6\sqrt{2}+2\sqrt{5}-4\sqrt{2})\\&\quad=(3\sqrt{5}-3\sqrt{2})(2\sqrt{5}+2\sqrt{2})=6(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})\\&\quad=6[(\sqrt{5})^2-(\sqrt{2})^2]=6(5-2)=18.\end{aligned}$$

SECTION TEST

QUESTIONS

1. What is the product of square roots of nonnegative numbers equal to?
2. What is $\sqrt{a^2}$ equal to for a positive number a?

EXERCISES

1. Simplify the following numerical expressions:

(a) $\sqrt{363}$; (b) $\sqrt{147}$; (c) $15\sqrt{1.04}-\frac{3}{5}\sqrt{5\frac{5}{9}}+6\sqrt{\frac{1}{18}}$; (d) $-(5\sqrt{0.02}-\sqrt{300})$.

2. Compute:

(a) $2(\sqrt{252}-\sqrt{175})(\sqrt{112}-\sqrt{63}-\sqrt{28})$; (b) $(\sqrt{6}-2\sqrt{15})\frac{\sqrt{3}}{3}+\sqrt{20}$.

3. Prove that for any real number a and any nonnegative number b the following equality holds true:

$$\sqrt{a^2b}=|a|\sqrt{b}.$$

4. Prove that for any real number a and any positive number b the following equality holds true:

$$\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}.$$

Chapter 12

QUADRATIC EQUATIONS

Sec. 12.1. The Notion of a Quadratic Equation

A *quadratic equation* is an equation the left-hand side of which is a standard-form polynomial of the second degree with respect to x, and the right-hand side is zero.

A quadratic equation is also called an *equation of second degree* or a *second-degree equation*.

The equations

$$2x^2 - 3x - 7 = 0; \quad x^2 - 4x = 0; \quad 2x^2 - 3 = 0,$$
$$x^2 - 11 = 0; \quad x^2 + 3x + 5 = 0$$

may serve as examples of quadratic equations.

The *general form* of a quadratic equation is

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b and c are given numbers ($a \neq 0$). The number a is called the coefficient of x^2, the number b the coefficient of x, the number c is the constant term of Equation (1), the term or part that does not contain x in any form. The expressions ax^2, bx, and c are called the terms of Equation (1). Thus, in the equation

$$2x^2 - 3x - 7 = 0$$

2 is the coefficient of x^2, (—3) the coefficient of x, (—7) the constant term; in the equation

$$x^2 - 3 = 0$$

1 is the coefficient of x^2, 0 the coefficient of x, (—3) the constant term.

The reader should recall that a root (or solution) of an equation is defined as a number which, when substituted for x, turn the equation into a true numerical equality.

For instance, the number 0 is the root of the equation

$$2x^2 - 7x = 0,$$

since if 0 is substituted for x, we get a true numerical equality

$$2 \cdot 0^2 - 7 \cdot 0 = 0.$$

To solve an equation means to find all of its roots.

In the next sections we shall show how to solve a quadratic equation, that is, how to find all of its roots.

When solving quadratic equations we shall have to multiply or divide both sides of an equation by a nonzero number and also to transpose terms from one side of an equation to the other. All this will result in a new equation equivalent to the original equation, that is, in a new equation having the same roots as the original equation.

SECTION TEST

QUESTIONS

1. What is a quadratic equation?
2. What is a root of an equation?
3. What does it mean to solve an equation?
4. What is a constant term of a quadratic equation?
5. What equations are called equivalent?

EXERCISES

1. Which of the following equations are quadratic:
(a) $x^2 + 4x - 5 = 0$; (b) $x^2 = 0$; (c) $2x^2 + 1 = 0$; (d) $4x - 5 = 0$?
2. Is the number $\left(-\frac{9}{16}\right)$ a root of the indicated equation:
(a) $2x^2 - 3x + 6 = 0$; (b) $16x^2 + 25x + 9 = 0$?
3. Are the following equations equivalent:
(a) $2x^2 - 4x + 6 = 0$ and $x^2 - 2x + 3 = 0$;
(b) $x^2 - 4x + 1 = 0$ and $x^2 + 1 = 4x$;
(c) $x^2 - 4 = 0$ and $x - 2 = 0$?

Sec. 12.2. Pure (Incomplete) Quadratic Equations

The quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

is called a *pure* (or *incomplete*) quadratic if $b = 0$ or $c = 0$.

In the present section we shall consider how to solve pure quadratic equations.

Example 1. Solve the equation

$$x^2 = 0. \tag{1}$$

We know that there is only one number, zero, whose square equals zero. Consequently, Equation (1) has the only one root $x_0 = 0$.

The pure quadratic equation in which $b = c = 0$, that is, the equation

$$ax^2 = 0 \qquad (a \neq 0)$$

is equivalent to Equation (1) and, consequently, also has a unique root $x_0 = 0$.

Example 2. Solve the equation

$$x^2 - 9 = 0. \tag{2}$$

Transposing the constant term to the right-hand side, we get the equation

$$x^2 = 9, \tag{3}$$

which is equivalent to Equation (2).

Thus, instead of solving Equation (2), we may solve Equation (3). The problem has reduced to finding all the numbers the square of each of which is equal to 9. As we know, there are two such numbers: 3 and (—3). Consequently, Equation (2) has two roots: $x_1 = 3$ and $x_2 = -3$. It has no other roots.

Example 3. Solve the equation

$$x^2 - 5 = 0. \tag{4}$$

This equation is equivalent to the equation

$$x^2 = 5.$$

Consequently, we have to find the numbers whose squares are equal to the number 5; there are only two such numbers: $\sqrt{5}$ and $-\sqrt{5}$.

Thus, Equation (4) has two roots: $x_1 = \sqrt{5}$ and $x_2 = -\sqrt{5}$, and there are no other roots.

Example 4. Solve the equation

$$x^2 + 7 = 0. \tag{5}$$

Obviously, this equation has no roots. Indeed, the square of any real number x_0 is nonnegative and, consequently, $x_0^2 + 7$ is a positive number. In other words, no real number x_0 can be a root of Equation (5). Thus, we have solved Equation (5), that is, we have shown that it has no real roots.

Later on, we shall introduce complex numbers. Then the reader will learn that the equation under consideration has two complex roots. But for the time being we know nothing about complex numbers and, therefore, we shall say that the given equation has no roots omitting the word "real".

If $b = 0$, then the pure quadratic equation has the form

$$ax^2 + c = 0 \qquad (a \neq 0). \tag{6}$$

This equation is equivalent to the equation

$$x^2 + \frac{c}{a} = 0. \tag{7}$$

It is clear that if $\frac{c}{a}$ is positive, then Equation (7), and, hence, Equation (6), have no roots (a in Example 4).

And if $\frac{c}{a}$ is a negative number, then, as in Example 3, Equation (7) will be equivalent to the equation

$$x^2 = -\frac{c}{a}. \tag{8}$$

Since the number $\left(-\frac{c}{a}\right)$ is positive, we have

$$-\frac{c}{a} = \left(\sqrt{\frac{c}{a}}\right)^2.$$

Consequently, Equation (8), and, hence, Equation (6) has two roots:

$$x_1 = \sqrt{-\frac{c}{a}} \quad \text{and} \quad x_2 = -\sqrt{-\frac{c}{a}},$$

and has no other roots.

For $c = 0$ Equation (6), as we know, has the unique solution $x = 0$.

Example 5. Solve the equation

$$x^2 - 3x = 0. \tag{9}$$

Let us rewrite this equation in the form

$$x(x - 3) = 0. \tag{10}$$

Obviously, this equation has two roots: $x_1 = 0$ and $x_2 = 3$. It has no other roots, since if we substitute any other number, different from 0 and 3, for x, then the left-hand side of Equation (10) will turn out to be equal to a number not equal to zero.

The pure quadratic equation, in which $c = 0$, has the form

$$ax^2 + bx = 0 \qquad (a \neq 0). \tag{11}$$

This equation is equivalent to the equation

$$\left(x + \frac{b}{a}\right) x = 0. \tag{12}$$

The same as in Example 5, it may be shown that Equation (12), and hence, Equation (11), has two roots:

$$x_1 = 0 \quad \text{and} \quad x_2 = -\frac{b}{a}$$

and these equations have no other roots.

The above examples show that a pure quadratic equation may have one or two roots, but it may also have no roots.

As we shall see in the next section, the considered examples are characteristic of quadratic equations, any of them has either two roots or one root or no roots at all.

SECTION TEST

QUESTIONS

1. What is a pure (incomplete) quadratic equation?
2. How many roots may a pure quadratic equation have?

EXERCISES

1. Solve the following equations:
(a) $-3x^2 = 0$; (b) $2x^2 + 3 = 0$; (c) $7x^2 - 1 = 0$;
(d) $8x^2 + 12 = 0$; (e) $3x^2 - 5 = 0$; (f) $3x^2 - 4x = 0$;
(g) $x^2 + 2x = 0$.

Sec. 12.3. Solving a General Quadratic Equation

In the present section we shall derive the formula for solving a general quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0). \tag{1}$$

Since $a \neq 0$, Equation (1) is equivalent to the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{2}$$

In Chapter 7 we considered the transformation of a second-degree polynomial called "separating a perfect square". Let us apply this transformation to the polynomial in the left-hand side of Equation (2):

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + 2 \cdot x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$$

$$= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

Let us introduce the number

$$D = b^2 - 4ac,$$

which is called the *discriminant* of Equation (1).

Obviously, Equation (2) is equivalent to the equation

$$\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} = 0. \tag{3}$$

Three cases are possible here:

Case 1. Let

$$D = b^2 - 4ac > 0.$$

Then the square root of the number D has sense and $D = (\sqrt{D})^2$. Therefore Equation (2) is equivalent to the equation

$$\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{D}}{2a}\right)^2=0.$$

Applying the formula for the difference between two squares to the left-hand side of this equation, we get the equivalent equation

$$\left(x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right)=0. \tag{4}$$

Setting

$$x_1=\frac{-b+\sqrt{D}}{2a},\quad x_2=\frac{-b-\sqrt{D}}{2a}, \tag{5}$$

we rewrite Equation (4) in the form

$$(x-x_1)(x-x_2)=0.$$

Obviously, this equation has the roots

$$x=x_1 \text{ and } x=x_2$$

and has no other roots. Indeed, if we substitute any number different from x_1 and x_2 for x, then the left-hand side turns into a number not equal to 0.

Thus, for $D>0$ Equation (1) has two roots: x_1 and x_2 computed by Formulas (5).

Case 2. Let

$$D=b^2-4ac=0.$$

Then Equation (3) will take the form

$$\left(x+\frac{b}{2a}\right)^2=0. \tag{6}$$

As it was established above, there is one number

$$x_0+\frac{b}{2a}=0,$$

satisfying Equation (6).

Hence, for $D=0$ Equation (1) has the only root

$$x_0=-\frac{b}{2a}. \tag{7}$$

However, it is convenient to regard that in the present case Equation (7) also has two, but equal (coincident) roots:

$$x_1=x_2=x_0=-\frac{b}{2a}.$$

Case 3. Let

$$D=b^2-4ac<0.$$

Then Equation (3) is equivalent to the equation

$$\left(x+\frac{b}{2a}\right)^2=\frac{D}{4a^2}. \tag{8}$$

Equation (8) has no real roots since the square of any real number cannot be a negative number. Hence, for $D<0$ Equation (1) has no roots.

Thus, quadratic equation (1) has no roots if its discriminant is negative; has the only root if the discriminant is equal to zero; and has two roots if the discriminant is positive.

Remark 1. If the discriminant of Equation (1) is positive, then Formulas (5) for the roots of this equation is often written as one formula

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{9}$$

Remark 2. If $D=0$, then Formula (9) remains true. In this case it yields one root

$$x_0=-\frac{b}{2a},$$

or, as we say, two equal (coincident) roots (or a double root):

$$x_1=x_2=x_0=-\frac{b}{2a}.$$

Henceforward, when solving quadratic equations, we shall not repeat the aforegoing reasoning, and shall use Formula (9).

Here are several examples.

Example 1. Solve the equation

$$x^2-x-12=0. \tag{10}$$

First, we compute the discriminant of the given equation:

$$D=b^2-4ac=(-1)^2-4\cdot1\cdot(-12)=49>0.$$

Hence, Equation (10) has two roots which are computed by the formula

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{1\pm\sqrt{49}}{2}=\frac{1\pm7}{2}.$$

Hence, Equation (10) has two roots: $x_1=4$, $x_2=-3$.

Example 2. Solve the equation

$$3x^2+2x-2=0. \tag{11}$$

We compute the discriminant of Equation (11):

$$D=b^2-4ac=2^2-4\cdot3\cdot(-2)=28>0.$$

Hence, Equation (11) has two roots, which are computed by the formula

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-2\pm\sqrt{28}}{2\cdot 3}=\frac{-2\pm 2\sqrt{7}}{2\cdot 3}=\frac{-1\pm\sqrt{7}}{3}.$$

Thus, Equation (11) has two roots:

$$x_1=\frac{-1+\sqrt{7}}{3},\quad x_2=\frac{-1-\sqrt{7}}{3}.$$

Example 3. Solve the equation

$$2x^2-4x+3=0. \tag{12}$$

We compute the discriminant of Equation (12):

$$D=b^2-4ac=(-4)^2-4\cdot 2\cdot 3=-8<0.$$

Hence, Equation (12) has no roots.

Example 4. Solve the equation

$$25x^2-30x+9=0. \tag{13}$$

We compute the discriminant of Equation (13):

$$D=b^2-4ac=(-30)^2-4\cdot 25\cdot 9=0.$$

Hence, Equation (13) has only one root which can be computed by the formula

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{30\pm 0}{2\cdot 25}=\frac{30}{50}=\frac{3}{5}.$$

Thus, Equation (13) has the only root $\frac{3}{5}$, or, which is the same, two equal roots (that is, a double root):

$$x_1=x_2=\frac{3}{5}.$$

SECTION TEST

QUESTIONS

1. What is the discriminant of a quadratic equation?
2. How many roots has a quadratic equation if its discriminant is: (a) negative, (b) positive, (c) zero?
3. What formula may be used to find the roots of a quadratic equation if its discriminant is nonnegative?

EXERCISES

1. Solve the following equations:
(a) $3x^2-7x+4=0$; (b) $x^2-7x+6=0$;
(c) $2x^2-7x+6=0$; (d) $25x^2+30x+9=0$.

Sec. 12.4. A Reduced Quadratic Equation

A quadratic equation with coefficient of x^2 equal to 1 is called a *reduced equation*.

The equations

$$x^2 - 2x + 7 = 0; \quad x^2 - 5 = 0; \quad x^2 = 0; \quad x^2 - 3x = 0$$

may serve as examples of reduced quadratic equations.

A reduced quadratic equation in general form is usually written as follows:

$$x^2 + px + q = 0, \tag{1}$$

where p and q are given numbers. The number p is the coefficient of x and q is a constant term.

Thus, Equation (1) may be regarded as a particular case of the general quadratic equation

$$ax^2 + bx + c = 0, \tag{2}$$

where

$$a = 1, \quad b = p, \quad c = q.$$

The discriminant of Equation (1) is equal to

$$D = b^2 - 4ac = p^2 - 4q. \tag{3}$$

Let $D > 0$. Then, as we know, Equation (1) has two roots which are computed by the formula

$$x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

It is usually written in the form

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}. \tag{4}$$

For $D = 0$ this formula is also used to compute equal roots. Namely, in this case

$$x_1 = x_2 = -\frac{p}{2}. \tag{5}$$

And if $D < 0$, then, as we know, Equation (1) has no (real) roots at all.

In the case of a general-form reduced equation, instead of the discriminant D, we usually consider the expression

$$\frac{D}{4} = \left(\frac{p}{2}\right)^2 - q,$$

which has the same sign as D.

We have shown that

(1) if $\left(\frac{p}{2}\right)^2 - q > 0$, then Equation (1) has two roots computed by Formula (4);

(2) if $\left(\frac{p}{2}\right)^2 - q = 0$, then Equation (1) has two equal (coinciding) roots computed by the same formula, or, which is the same, by Formula (5);

(3) if $\left(\frac{p}{2}\right)^2 - q < 0$, then Equation (1) has no real roots.

Example. Solve the equation

$$x^2 - 8x + 7 = 0. \tag{6}$$

We compute:

$$\left(\frac{p}{2}\right)^2 - q = (-4)^2 - 7 = 9 > 0;$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} = 4 \pm \sqrt{9} = 4 \pm 3.$$

Consequently, Equation (6) has two roots: $x_1 = 7$, $x_2 = 1$.

SECTION TEST

QUESTIONS

1. What is a reduced quadratic equation?
2. What formula may be used to find the roots of a reduced quadratic equation if its discriminant is nonnegative?

EXERCISES

1. Solve the following equations:
(a) $x^2 - 8x - 9 = 0$; (b) $x^2 - 6x + 9 = 0$;
(c) $x^2 - 16x + 65 = 0$.

Sec. 12.5. Applying Quadratic Equations to Solving Problems

Let us consider several problems.

Problem 1. The sum of the squares of two consecutive natural numbers is greater than the product of these numbers by 57. Find these numbers.

Solution. Denote the smaller of the desired numbers by x, then the greater number will be $(x + 1)$. By the hypothesis,

$$x^2 + (x + 1)^2 = x(x + 1) + 57. \tag{1}$$

Transposing all the terms of Equation (1) to the left side and collecting like terms, we get the equation

$$x^2 + x - 56 = 0, \tag{2}$$

which is equivalent to Equation (1). The roots of the last equation are computed by the formula

$$x_{1,2} = \frac{-1 \pm \sqrt{1+4 \cdot 56}}{2} = \frac{-1 \pm 15}{2}.$$

Consequently, Equation (2) and its equivalent, Equation (1), have the roots

$$x_1 = 7, \quad x_2 = -8. \tag{3}$$

Since we have to find the natural number x satisfying Equation (1), of these two roots only one satisfies the conditions of the problem: $x_1 = 7$.

Answer. The sought-for numbers are 7 and 8.

Problem 2. The price of an article, which first was equal to 25 roubles, after two successive cuts by the same percentage is now 20 roubles and 25 copecks. By how many percent was the price reduced each time?

Solution. Let the price be reduced each time by $x\%$. This means that after the first cut the price became lower by $\frac{x}{100}$ parts of the original cost, that is, the price of the article amounted to $25\left(1 - \frac{x}{100}\right)$ roubles. After the second cut the price became $\left(1 - \frac{x}{100}\right)$ parts of the cost after the first cut, that is the final price amounted to $25\left(1 - \frac{x}{100}\right)\left(1 - \frac{x}{100}\right)$ roubles. But, by the hypothesis, after the two price cuts the article costed 20 roubles and 25 copecks. Hence,

$$25\left(1 - \frac{x}{100}\right)^2 = 20.25. \tag{4}$$

From this equation we have to find x. Transposing all the terms of Equation (4) to the left-hand side and removing the parentheses we get the equation

$$\frac{x^2}{400} - \frac{x}{2} + 4.75 = 0. \tag{5}$$

Multiplying this equation by 400, we get the equation

$$x^2 - 200x + 1900 = 0. \tag{6}$$

Solving Equation (6), we find its roots: $x_1 = 10$, $x_2 = 190$. Since Equations (5) and (4), (6) and (5) are equivalent, Equation (4) also has the roots: $x_1 = 10$ and $x_2 = 190$. Since it is impossible to reduce the price by 190%, only $x_1 = 10$ satisfies the conditions of the problem.

Answer. The price was reduced by 10% each time.

SECTION TEST

EXERCISES

1. The product of two successive natural numbers exceeds their sum by 109. Find these numbers.

2. Find the perimeter of the rectangle whose length is 4 cm greater than its width, and the area is equal to 60 cm^2.

Chapter 13

RATIONAL EQUATIONS

In the present chapter we shall consider several types of equations whose solution is reduced to solving equations of the first and second degree.

These equations may be called *rational equations* since their both sides are rational expressions of an unknown x.

Sec. 13.1. Quartic (or Biquadratic) Equations

An equation of the form

$$ax^4 + bx^2 + c = 0, \tag{1}$$

where a, b, and c are given numbers ($a \neq 0$), and x is an unknown, is called a *quartic* (or *biquadratic*) equation.

To solve Equation (1), we introduce a new unknown by means of the equality

$$y = x^2. \tag{2}$$

Then Equation (1) turns into a quadratic equation

$$ay^2 + by + c = 0 \tag{3}$$

with respect to the unknown y.

If Equation (3) has no roots, then, obviously, the given equation (1) also has no roots.

And if Equation (3) has roots, then, substituting them into Equation (2) for y, we get equations with respect to x. The solutions of the obtained equations, if any, are just the solutions of Equation (1). Obviously, Equation (1) has no other solutions.

Consider several quadratic equations:

$$x^4 - 4x^2 + 3 = 0; \tag{4}$$

$$x^4 - 2x^2 - 2 = 0; \tag{5}$$

$$2x^4 - 3x^2 + 5 = 0; \tag{6}$$

$$9x^4 - 6x^2 + 1 = 0; \tag{7}$$

$$x^4 + 10x^2 + 25 = 0. \tag{8}$$

With substitution $y = x^2$ Equation (4) turns into the quadratic equation

$$y^2 - 4y + 3 = 0.$$

Its discriminant

$$D = b^2 - 4ac = 16 - 12 = 4 > 0,$$

and, consequently it has two real roots:

$$y_{1,2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2},$$

that is,

$$y_1 = 1 \text{ and } y_2 = 3.$$

Substituting these numbers for y into Equation (2), we get equations with respect to x:

$$1 = x^2 \text{ and } 3 = x^2.$$

Solving them, we obtain four roots of Equation (4):

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = \sqrt{3}, \quad x_4 = -\sqrt{3}.$$

Obviously, Equation (4) has no other roots.

After the substitution $y = x^2$, Equation (5) turns into the quadratic equation

$$y^2 - 2y - 2 = 0.$$

Its discriminant

$$D = b^2 - 4ac = 4 + 8 = 12 > 0,$$

and, therefore it has two real roots determined by the formula

$$y_{1,2} = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3},$$

that is,

$$y_1 = 1 + \sqrt{3}, \quad y_2 = 1 - \sqrt{3}.$$

Substituting y_1 for y into Equality (2), we get the equation

$$1 + \sqrt{3} = x^2,$$

whence we get two roots of Equation (5):

$$x_1 = \sqrt{1 + \sqrt{3}}. \quad x_2 = -\sqrt{1 + \sqrt{3}}.$$

Substituting y_2 into Equality (2), we get the equation

$$1 - \sqrt{3} = x^2,$$

which has no real roots since $1 - \sqrt{3} < 0$.

Thus, Equation (5) has two real roots x_1 and x_2 found above, and has no other roots.

After the substitution $y = x^2$, Equation (6) turns into the quadratic equation

$$2y^2 - 3y + 5 = 0.$$

Its discriminant

$$D = b^2 - 4ac = 9 - 40 = -31 < 0.$$

Consequently, it has no real roots. But then Equation (6) also has no real roots.

With the substitution $y = x^2$, Equation (7) turns into the quadratic equation

$$9y^2 - 6y + 1 = 0$$

with the discriminant

$$D = b^2 - 4ac = 36 - 36 = 0.$$

Thus, it has only one root

$$y_0 = \frac{6 \pm 0}{18} = \frac{6}{18} = \frac{1}{3}.$$

Solving the equation

$$\frac{1}{3} = x^2,$$

we get two roots of Equation (7):

$$x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}.$$

Equation (7) has no other roots.

With the substitution $y = x^2$, Equation (8) turns into the quadratic equation

$$y^2 + 10y + 25 = 0$$

with the discriminant

$$D = b^2 - 4ac = 100 - 100 = 0.$$

Thus, it has the unique root

$$y_0 = \frac{-10 \pm 0}{2} = -5.$$

Substituting y_0 into Equality (2), we get the equation

$$x^2 = -5,$$

which has no real roots. Hence, Equation (8) also has no roots.

Remark. As is seen from the considered examples, a quartic equation may have not more than four real roots, but may have none. Later on, when studying complex numbers, we shall learn that a quartic equation has, generally speaking, four complex roots. However, there are cases when it has less than four roots, but then we regard that some roots coincide.

SECTION TEST

QUESTIONS

1. What equation is called quartic (or biquadratic)?
2. How is a quartic equation solved?

EXERCISES

1. Solve the following equations:
(a) $x^4 - 5x^2 + 6 = 0$; (b) $5x^4 + 8x^2 + 3 = 0$; (c) $4x^4 - 5x^2 - 6 = 0$;
(d) $9x^4 - 24x^2 + 16 = 0$; (e) $25x^4 + 30x^2 + 9 = 0$;
(f) $7x^4 - 9x^2 + 3 = 0$.

Sec. 13.2. Factorable Equations

Example 1. Solve the equation

$$(x^2 - 5x + 6)(x^2 + x - 2) = 0. \tag{1}$$

If the number x_0 is a root of Equation (1), then substituting x_0 for x into Equation (1), we get a true numerical equality

$$(x_0^2 - 5x_0 + 6)(x_0^2 + x_0 - 2) = 0. \tag{2}$$

But the product of two numbers is equal to zero if and only if one of the factors is equal to zero or if both of them equal zero. Therefore it follows from (2) that x_0 is a root of at least one of the equations

$$x^2 - 5x + 6 = 0, \tag{3}$$

$$x^2 + x - 2 = 0. \tag{4}$$

Thus, any root of Equation (1) is a root of at least one of Equations (3) and (4). It is obvious that any root of either of Equations (3) or (4) is a root of Equation (1).

In this sense, Equation (1) is said to decompose into Equations (3) and (4).

Equation (3) has the roots $x_1 = 2$ and $x_2 = 3$, and Equation (4) has the roots $x_3 = -2$ and $x_4 = 1$. Consequently, Equation (1) has the following four roots:

$$x_1 = 2, \quad x_2 = 3, \quad x_3 = -2, \quad x_4 = 1$$

and has no other roots.

Example 2. Solve the equation

$$x^3 - 1 = 0. \tag{5}$$

Equation (5) decomposes into two equations:

$$x - 1 = 0, \tag{6}$$
$$x^2 + x + 1 = 0, \tag{7}$$

since

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

Equation (6) has, obviously, the root $x_1 = 1$, while Equation (7) has no real roots. Consequently, Equation (5) has only one root

$$x_1 = 1.$$

Example 3. Solve the equation

$$x^6 - 1 = 0. \tag{8}$$

Equation (8) decomposes into three equations:

$$x - 1 = 0, \tag{9}$$
$$x + 1 = 0, \tag{10}$$
$$x^4 + x^2 + 1 = 0, \tag{11}$$

since

$$x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1) = (x - 1)(x + 1)(x^4 + x^2 + 1).$$

Equations (9) and (10) have the respective roots

$$x_1 = 1 \quad \text{and} \quad x_2 = -1.$$

Whereas Equation (11) has no real roots. Indeed, Equation (11) is quartic. The substitution

$$y = x^2$$

leads it to a quadratic equation

$$y^2 + y + 1 = 0,$$

which has no real roots. Consequently, Equation (8) has two roots:

$$x_1 = 1 \quad \text{and} \quad x_2 = -1.$$

Example 4. Solve the equation

$$x^3 - 2x^2 - 3x = 0. \tag{12}$$

Since

$$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3).$$

Equation (12) decomposes into two equations:

$$x^2 - 2x - 3 = 0 \text{ and } x = 0.$$

The first of them has two roots:

$$x_{1,2}=1\pm\sqrt{1+3}=1\pm 2,$$

but then Equation (12) has three roots:

$$x_1=3,\quad x_2=-1,\quad x_3=0.$$

SECTION TEST

QUESTIONS

1. How are factorable equations solved?

EXERCISES

1. Solve the following equations:
(a) $(x^2-2x+5)(x^2-3)(x+1)=0$; (b) $(x^4+x^2-2)(x^2-9)=0$;
(c) $x^5+5x^3+6x=0$.

Sec. 13.3. Simplest Rational Equations

Example 1. Solve the equation

$$\frac{x^2+4x-21}{x^2-x-3}=0. \tag{1}$$

We know that the root of an equation with respect to the unknown x is defined as a number x_0 which, being substituted for x, turns the equation into a true numerical equality. Therefore, if x_0 is a root of Equation (1), then the expression

$$\frac{x_0^2+4x_0-21}{x_0^2-x_0-3}$$

is a numerical expression equal to zero. But then the denominator of this expression must not be equal to zero, while the numerator must equal zero.

Thus, to solve Equation (1), we must find the roots of the equation

$$x^2+4x-21=0 \tag{2}$$

and substitute them into the denominator of the left-hand member of Equation (1). Those of them which do not turn the denominator into zero are the roots of Equation (1). This equation has no other roots.

The discriminant of the quadratic equation (2) is positive and, consequently, Equation (2) has two roots determined by the formula

$$x_{1,2}=-2\pm\sqrt{4+21}=-2\pm 5,$$

that is, $x_1=3$, $x_2=-7$.

Substituting these numbers into the denominator of the left-hand side of Equation (1), we get

$$x_1^2 - x_1 - 3 = 9 - 3 - 3 = 3 \neq 0,$$
$$x_2^2 - x_2 - 3 = 49 + 7 - 3 = 53 \neq 0.$$

This shows that the numbers

$$x_1 = 3, \quad x_2 = -7$$

are the roots of Equation (1), this equation has no other roots.

Example 2. Solve the equation

$$\frac{x^2 - x - 2}{x^3 - 2x^2 - 3x} = 0. \tag{3}$$

Reasoning as in Example 1, we first solve the equation

$$x^2 - x - 2 = 0.$$

It has two roots:

$$x_1 = 2, \quad x_2 = -1.$$

Substituting them into the denominator of the left-hand member of Equation (3), we get:

$$x_1^3 - 2x_1^2 - 3x_1 = 2^3 - 2 \cdot 2^2 - 3 \cdot 2 = -6 \neq 0,$$
$$x_2^3 - 2x_2^2 - 3x_2 = (-1)^3 - 2 \cdot (-1)^2 - 3 \cdot (-1) = 0.$$

Consequently, Equation (3) has the only root $x_1 = 2$.

Example 3. Solve the equation

$$\frac{2x - 3}{4x^4 + 4x^3 - 15x^2 + 2x - 3} = 0. \tag{4}$$

We first solve the equation

$$2x - 3 = 0.$$

It has the only root

$$x_1 = \frac{3}{2}.$$

Since

$$4x_1^4 + 4x_1^3 - 15x_1^2 + 2x_1 - 3$$
$$= 4 \cdot \left(\frac{3}{2}\right)^4 + 4 \cdot \left(\frac{3}{2}\right)^3 - 15 \cdot \left(\frac{3}{2}\right)^2 + 2 \cdot \frac{3}{2} - 3 = 0.$$

Equation (4) has no roots.

Example 4. Solve the equation

$$\frac{x^2 + x + 1}{x - 3} = 0. \tag{5}$$

Equation (5) has no roots since the equation

$$x^2 + x + 1 = 0$$

has no roots.

SECTION TEST

QUESTIONS

1. How is it possible to solve an equation one member of which is zero and the other is a rational fraction?

EXERCISES

1. Solve the following equations:

(a) $\frac{x^2+5x+6}{x^2+x-2}=0$; (b) $\frac{x^6-1}{x^3-1}=0$; (c) $\frac{x^2-1}{(x^2-3x+2)(x^2-2x-3)}=0$.

Sec. 13.4. Rational Equations

Example 1. Solve the equation

$$2-\frac{x+1}{x-1}=0. \tag{1}$$

The left-hand side of this equation is defined on the set of all real numbers, except $x = 1$. Consequently, the roots of this equation should be sought only among these values of x.

Applying the rule for subtracting algebraic fractions to the left-hand side of Equation (1), we get

$$2-\frac{x+1}{x-1}=\frac{2(x-1)-(x+1)}{x-1}=\frac{x-3}{x-1}. \tag{2}$$

For any number $x_0 \neq 1$ the numerical values of both sides of Equality (2) are equal. In particular, if a certain number x_0 turns one side of Equality (2) into zero, then the other side of this equality is also turned into zero by this number. And this means that Equation (1) is equivalent to the equation

$$\frac{x-3}{x-1}=0. \tag{3}$$

We know how to solve Equation (3) (see the preceding section). For this purpose, we first solve the equation

$$x - 3 = 0.$$

It has only one root $x_0 = 3$. In this case, the number $x_0 = 3$ does not turn into zero the denominator of the fraction in the left-hand side of Equation (3):

$$x_0 - 1 = 3 - 1 = 2 \neq 0.$$

Therefore Equation (3) has the only root $x_0 = 3$. Hence, original equation (1) also has only one root

$$x_0 = 3.$$

Example 2. Solve the equation

$$\frac{x-1}{x+2}-\frac{x-4}{x-3}=-1. \tag{4}$$

Transposing the right-hand side of Equation (4) to the left, we get the equation

$$\frac{x-1}{x+2}-\frac{x-4}{x-3}+1=0, \tag{5}$$

which is equivalent to Equation (4).

Let us apply the rules for adding and subtracting algebraic fractions to the left-hand side of Equation (5):

$$\frac{x-1}{x+2}-\frac{x-4}{x-3}+1 = \frac{(x-1)(x-3)-(x-4)(x+2)+(x+2)(x-3)}{(x+2)(x-3)}=\frac{x^2-3x+5}{(x+2)(x-3)}.$$

Reasoning as in Example 1, we obtain that Equation (5) is equivalent to the equation

$$\frac{x^2-3x+5}{(x+2)(x-3)}=0. \tag{6}$$

To solve Equation (6), we have first to solve the equation

$$x^2-3x+5=0.$$

Since its discriminant

$$D=b^2-4ac=(-3)^2-4\cdot1\cdot5=-11<0,$$

it has no roots.

Hence the original equation (4) has no roots.

The following rule follows from the aforegoing examples: to solve a rational equation, do the following: transpose all the terms to the left-hand side, then, applying the rules for adding and subtracting algebraic fractions, write the left-hand member as an algebraic fraction, and, finally, solve the obtained equation using the techniques described in Sec. 13.3.

Remark. Violation of this rule may result in lost roots or in acquiring extraneous roots.

For instance, if we apply the above rule to the equation

$$\frac{(x-2)(x-3)}{(x-3)}=1, \tag{7}$$

then we get the equivalent equation

$$\frac{(x-3)^2}{(x-3)}=0. \tag{8}$$

It has no roots. Consequently, Equation (7) also has no roots. However, if despite the rule, we reduced the fraction in the left-hand side of Equation (7) by $(x - 3)$, we would get the equation

$$x - 2 = 1, \tag{9}$$

which has the root $x = 3$. But $x = 3$ is not the root of Equation (7) since for $x = 3$ the left-hand side turns into an expression having no sense.

Hence, using such "method of solution", we have obtained an extraneous root for Equation (7).

Suppose we are first given Equation (9), and despite the rule, we multiply the numerator and denominator of the fraction $\frac{x-2}{1}$ by a nonzero polynomial $(x - 3)$, then we shall arrive at Equation (7) which has no roots. Hence, making use of such "method of solution", we have lost the root $x_0 = 3$ of Equation (9).

SECTION TEST

QUESTIONS

1. How do we solve rational equations?

EXERCISES

1. Solve the following equations:

(a) $\frac{1}{x^2+x+2} = \frac{1}{2x^2+3x-6}$; (b) $\frac{1}{x-1} + \frac{1}{(x-1)(x+2)} = 1$;

(c) $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x+3} - \frac{1}{x+4}$.

Sec. 13.5. Worked Problems

Problem 1. A steamer pushed off the landing-stage A and went downstream 60 km to reach the mouth of a tributary falling into the river. Then the steamer went up the tributary and reached the landing-stage B, 20 km distant from the mouth. It took the steamer 7 hours to cover the entire path from A to B. The rate of flow of both the river and tributary is equal to 1 km/h. Find the speed of the steamer in stagnant water.

Solution. Let us denote by x km/h the speed of the steamer in stagnant water. Then the steamer went downstream with a speed of $(x + 1)$ km/h and reached the mouth of the tributary in $\frac{60}{x+1}$ hours. The steamer went up the tributary with a speed $(x - 1)$ km/h

and reached the landing-stage B in $\frac{20}{x-1}$ hours. It took the steamer 7 hours to cover the entire path. Hence,

$$\frac{60}{x+1}+\frac{20}{x-1}=7. \tag{1}$$

Thus, we have obtained a rational equation.

Transposing all of its terms to the left-hand side and applying the rules for adding and subtracting fractions to the left-hand side of the obtained equation, we get the equation

$$\frac{7x^2-80x+33}{(x+1)(x-1)}=0, \tag{2}$$

which is equivalent to Equation (1).

The equation $7x^2 - 80x + 33 = 0$ has two roots: $x_1 = 11$ and $x_2 = \frac{3}{7}$.

Both of them do not turn into zero the denominator of the left-hand member of Equation (2), and therefore both of them are the roots of Equation (2), and hence, of Equation (1).

Thus, Equation (1) has two roots: $x_1 = 11$ and $x_2 = \frac{3}{7}$. But, according to the conditions of the problem, the speed of the steamer cannot be less than 1 km/h, since the steamer went up the tributary (that is, against the stream) whose rate of flow is equal to 1 km/h. Consequently, the condition of the problem is satisfied only by $x_1 = 11$.

Answer. The speed of the steamer in stagnant water is 11 km/h.

Problem 2. The operator of a trench excavator got an assignment to dig two trenches of equal depth at different sections of the building site. The excavator dug the first trench 5 m long, then drove to the second section and dug the second trench 3 m long. The time spend on digging the first trench is by 1 hour and 12 minutes less than the time spent on moving the excavator to a new place and digging the second trench. If the digging capacity of the excavator were one-fourth the actual capacity, then the time required for digging the first trench would be equal to the time necessary for changing the place of operation. Determine the hourly capacity of the excavator (that is, the length of the trench dug per hour).

Solution. Let us denote by x metres the length of the trench dug per hour. Then it took the excavator $\frac{5}{x}$ hours to dig the first trench and $\frac{3}{x}$ hours to dig the second trench. By the hypothesis, the time spent by the excavator to move to the other place of operation is four times the time spent by the excavator on digging the first trench, that is, it is equal to $\frac{20}{x}$ hours. Since the time during which the

excavator dug the first trench is by $1\frac{12}{60}$ hours less than the time spent on changing the place and digging the second trench, we get the equation

$$\frac{5}{x}+1\frac{12}{60}=\frac{20}{x}+\frac{3}{x}. \tag{3}$$

Transposing all the terms to the left-hand side and applying the rules for adding and subtracting common fractions, we get the equation

$$\frac{6x-90}{5x}=0, \tag{4}$$

which is equivalent to Equation (3). Equation (4) has the only root: $x_0 = 15$. Consequently, the length of the trench dug per hour is equal to 15 metres.

SECTION TEST

EXERCISES

1. The distance between the landing-stages A and B is 48 km. Pushing off the landing-stage A at 9 o'clock in the morning a steamer went downstream and reached the landing-stage B. An hour later the steamer set out for the return voyage and arrived at A at five o'clock in the evening. The rate of flow of the river is equal to 2 km/h. Find the speed of the steamer in stagnant water if it is known that on the way from A to B and from B to A it was the same.

2. A drilling rig made a borehole 1800 m deep at the point A, then the rig was moved to the point B, where another borehole 750 m deep was made. It took one month more to drill the first hole than to remove the rig and drill the second hole. If the drilling rate for the hole at the point A had been twice the actual rate, then the time spent drilling it would have been equal to the time required per month to move the rig from A to B. Determine the drilling rate per month (in metres).

Chapter 14

SYSTEMS OF LINEAR EQUATIONS

Sec. 14.1. A Linear Equation in Two Unknowns

The equation

$$ax + by + c = 0, \tag{1}$$

where a, b, and c are given numbers, and x and y are unknowns, is called a *linear equation in two unknowns.*

If one of the numbers a and b is not equal to zero, then Equation (1) is said to be a first-degree equation in two unknowns x and y since then its left-hand side is a polynomial of the first degree with respect to x and y.

The number a is called the coefficient of x, and the number b the coefficient of y.

The expressions

$$ax; \quad by; \quad c$$

are the terms of Equation (1), c being its constant term.

A pair of numbers (x_0, y_0) is a solution of Equation (1) if these numbers satisfy Equation (1), that is, if, when substituting x_0 for x and y_0 for y, Equation (1) turns into a true numerical equality:

$$ax_0 + by_0 + c = 0.$$

A linear equation in two unknowns may be exemplified by the equation

$$2x - 3y + 3 = 0. \tag{2}$$

In this equation

$$a = 2, \quad b = -3, \quad c = 3.$$

It is readily seen that Equation (2) has infinitely many solutions. If x is replaced by any number, then we obtain a first-degree equation in one unknown y, and on solving it we get a pair of numbers (x, y) which will be a solution of Equation (2).

For instance, setting $x = 0$, we get the equation

$$-3y + 3 = 0,$$

from which it follows that $y = 1$, and we obtain a solution of Equation (2) that is, the pair of numbers (0, 1). Setting $x = 1$, we get the equation $2-3y + 3 = 0$, whence $y = \frac{5}{3}$, and, consequently, the pair of numbers $\left(1, \frac{5}{3}\right)$ is a solution o Equation (2).

The general linear equation (1), where a, b and c are given numbers, both numbers a and b being different from zero, also has infinitely many solutions. We may assign an arbitrary value of x, compute the corresponding value of y, and obtain a solution (x, y). It turns out, that to each number x there corresponds its own solution (x, y) of Equation (1).

Suppose that in Equation (1)

$$a = b = 0, \quad c \neq 0,$$

then it will take the form

$$0 \cdot x + 0 \cdot y + c = 0,$$

and, obviously, no pair of numbers (x, y) satisfies Equation (1), that is, Equation (1) has no solutions.

And if $a = b = c = 0$, then Equation (1) is satisfied by all possible pairs of numbers (x, y), where x and y are any numbers.

Let us now suppose that in Equation (1)

$$a \neq 0, \quad b = 0,$$

then it has the form

$$ax + 0 \cdot y + c = 0. \tag{3}$$

Substituting any number for y into Equation (3), we get the first-degree equation in one unknown x

$$ax + c = 0,$$

which has a unique solution $x = -\frac{c}{a}$. Then all possible solutions of Equation (3) have the form $\left(-\frac{c}{a}, y\right)$, where y is any real number. And this means that Equation (3) has infinitely many solutions.

And if $a = 0$, $b \neq 0$, then Equation (1) has the form

$$0 \cdot x + by + c = 0. \tag{4}$$

Substituting any number for x into Equation (4), we get the equation

$$by + c = 0,$$

which has a unique solution $y = -\frac{c}{b}$. Consequently, all possible solutions of Equation (4) have the form $\left(x, -\frac{c}{b}\right)$, where x is any real number. Hence, Equation (4) has infinitely many solutions.

SECTION TEST

QUESTIONS

1. What is a linear equation in two unknowns?
2. What is a first-degree equation in two unknowns?
3. What is a solution of a linear equation in two unknowns?
4. How many solutions has the equation $ax + by + c = 0$ for a and b different from zero?

EXERCISES

1. Write five examples of linear equations in two unknowns.
2. Is the pair of numbers (1, 1) a solution of the equations from Exercise 1?
3. Write five solutions of the equation $x - y + 1 = 0$.
4. How many solutions has the equation $x - y + 1 = 0$?

Sec. 14.2. A System of Two Linear Equations in Two Unknowns

Let there be given two linear equations in two unknowns x and y:

$$ax + by + c = 0 \text{ and } a_1x + b_1y + c_1 = 0, \tag{1}$$

where a, b, c, a_1, b_1, c_1 are given numbers. We say that there is given a system of two linear equations in two unknowns x and y if it is

required to find all pairs of numbers (x, y) which are simultaneously the solutions of both the first and second equations (1). One equation is usually placed below the other, the two of them being united with a brace:

$$\begin{cases} ax+by+c=0, \\ a_1x+b_1y+c_1=0. \end{cases} \tag{2}$$

If both equations of System (2) are first-degree equations, then (2) is called a system of first-degree equations.

A solution of System (2) is defined as a pair of numbers (x, y) which is a solution of both equations of the system.

To solve a system means to find all of its solutions or to prove that the given system has no solutions.

Given below are examples of systems of two linear equations in two unknowns:

$$\begin{cases} x+y+3=0, \\ x-y-1=0; \end{cases} \tag{3}$$

$$\begin{cases} 2x-3y+1=0, \\ x+2y+4=0; \end{cases} \tag{4}$$

$$\begin{cases} 3x+0\cdot y+1=0, \\ 2x+y-5=0; \end{cases} \tag{5}$$

$$\begin{cases} 2x+0\cdot y+5=0, \\ 3x+0\cdot y+2=0; \end{cases} \tag{6}$$

$$\begin{cases} 5x+0\cdot y-1=0, \\ 0\cdot x+3y+2=0; \end{cases} \tag{7}$$

$$\begin{cases} 2x+3y-4=0, \\ 0\cdot x+0\cdot y+2=0; \end{cases} \tag{8}$$

$$\begin{cases} 7x-4y+5=0, \\ 0\cdot x+0\cdot y+0=0; \end{cases} \tag{9}$$

The terms $0\cdot x$ and $0\cdot y$ are usually omitted. Then Systems (5), (6), and (7) are written in such a form:

$$\begin{cases} 3x+1=0, \\ 2x+y-5=0; \end{cases} \tag{5'}$$

$$\begin{cases} 2x+5=0, \\ 3x+2=0; \end{cases} \tag{6'}$$

$$\begin{cases} 5x-1=0, \\ 3y+2=0. \end{cases} \tag{7'}$$

Obviously, System (7) is satisfied only by the pair of numbers $x=\frac{1}{5}$, $y=-\frac{2}{3}$, that is, System (7) has the only solution $\left(\frac{1}{5}, -\frac{2}{3}\right)$.

It is also obvious that System (8) has no solutions, since its second equation has no solution.

Let us now solve System (9). The second equation of this system is satisfied by all possible pairs (x, y), where x and y are any numbers. Consequently, solutions of System (9) are those pairs (x, y) which satisfy the first equation of the system. Since the first equation is a first-degree equation, it has infinitely many solutions. Setting $x = \alpha$, where α is an arbitrary number, we get the equation of the first degree with respect to y:

$$7\alpha - 4y + 5 = 0,$$

whence we find that $y=\frac{7}{4}\alpha+\frac{5}{4}$.

Hence, all possible solutions of the first equation of the system, and hence, of the entire system (9) have the form

$$\left(\alpha, \frac{7}{4}\alpha+\frac{5}{4}\right),$$

where α is an arbitrary number.

Let us also solve System (5). From the first equation it follows that

$$x=-\frac{1}{3}.$$

Substituting this number for x into the second equation, we get

$$2\cdot\left(-\frac{1}{3}\right)+y-5=0,$$

whence $y=\frac{17}{3}$.

Consequently, the pair of numbers $\left(-\frac{1}{3}, \frac{17}{3}\right)$ is the only solution of System (5).

Note that the formal solution of this system can be as follows.

Let a pair of numbers (x, y) be its solution. Let us assume that these numbers are substituted into the equations of System (5). Then the equations of the system will turn into true numerical equalities. But then from the first numerical equality we find that $x = -\frac{1}{3}$. Substituting this value of x into the second equality, we get

$$2\cdot\left(-\frac{1}{3}\right)+y-5=0,$$

whence we find that $y = \frac{17}{3}$.

Thus, we have shown that if System (5) has a solution, then this solution can be represented only by the pair of numbers $\left(-\frac{1}{3}, \frac{17}{3}\right)$. Substituting these numbers for x and y into the equations of System (5), we make sure that they satisfy these equations. Consequently, System (5) has the only solution $\left(-\frac{1}{3}, \frac{17}{3}\right)$.

The next two sections deal with the methods of solving the system

$$\begin{cases} ax+by+c=0, \\ a_1x+b_1y+c_1=0. \end{cases}$$

where a, b, a_1, b_1 are different from zero.

SECTION TEST

QUESTIONS

1. What is a system of two linear equations in two unknowns?
2. What is a system of two first-degree equations in two unknowns?
3. What is a solution of a system of two linear equations in two unknowns?
4. What does it mean to solve a system of two linear equations in two unknowns?

EXERCISES

1. Write five examples of systems of two linear equations in two unknowns.
2. Find out whether the pair of numbers (−3, 1) is a solution of the following system of equations:

(a) $\begin{cases} x+y-3=0, \\ 2x-3y-1=0; \end{cases}$ (b) $\begin{cases} x-y+4=0, \\ 3x+4y+5=0. \end{cases}$

3. Solve the following systems of equations:

(a) $\begin{cases} 2x+1=0, \\ 3y+5=0; \end{cases}$ (b) $\begin{cases} 2x+4=0, \\ x+2y-1=0; \end{cases}$

(c) $\begin{cases} x-3y+1=0, \\ y-7=0; \end{cases}$ (d) $\begin{cases} 2y-3=0, \\ 5+y=0. \end{cases}$

Sec. 14.3. The Substitution Method

Example 1. Solve the system of equations

$$\begin{cases} 2x-3y+1=0, \\ 3x+4y-2=0. \end{cases} \tag{1}$$

Let a pair of numbers (x, y) be a solution of System (1). Suppose these numbers are substituted into the equations of System (1). Then these equations will turn into true numerical equalities. From the first numerical equality we express y in terms of x:

$$y=\frac{2}{3}x+\frac{1}{3}, \tag{2}$$

and substitute the obtained expression $\left(\frac{2}{3}x+\frac{1}{3}\right)$ into the second numerical equality:

$$3x+4\cdot\left(\frac{2}{3}x+\frac{1}{3}\right)-2=0. \tag{3}$$

From Equality (3) we find $x=\frac{2}{17}$. Substituting the found value of x into Equality (2), we get $y=\frac{7}{17}$.

Thus, we have shown that if System (1) has a solution, then it can be represented by the pair of numbers $x=\frac{2}{17}$, $y=\frac{7}{17}$.

Substituting these numbers into the equations of System (1), we make sure that they satisfy these equations. Consequently, System (1) has the only solution $\left(\frac{2}{17}, \frac{7}{17}\right)$.

Example 2. Solve the system of equations

$$\begin{cases} x+y+1=0, \\ 2x+2y+3=0. \end{cases} \tag{4}$$

Let a pair of numbers (x, y) be a solution of System (4). Substituting these numbers into the equations of System (4), we turn them into true numerical equalities. Finding from the first equality

$$y = -x - 1$$

and substituting it into the second, we get

$$2x + 2(-x - 1) + 3 = 0$$

or, after simplification,

$$1 = 0.$$

We have obtained a contradiction. It shows that our supposition that System (4) has a solution is untrue. Hence, System (4) has no solution.

Example 3. Solve the system of equations

$$\begin{cases} x+y+1=0, \\ 2x+2y+2=0. \end{cases} \tag{5}$$

Let a pair of numbers (x, y) be a solution of System (5). Substituting the numbers into the equations of System (5), we shall turn them into true numerical equalities. Finding from the first equality

$$y = -x - 1 \tag{6}$$

and substituting it into the second, we get

$$2x + 2(-x - 1) + 2 = 0$$

or, after simplification,

$$0 = 0.$$

This equality shows that x may be any number which will be denoted by $\alpha : x = \alpha$. Substituting α for x into Equality (6), we get

$$y = -\alpha - 1.$$

It is easily seen that for any α the pair of numbers $x = \alpha$, $y = -\alpha - 1$ satisfies System (5). Consequently, System (5) has infinitely many solutions, all of them have the form

$$(\alpha, -\alpha - 1),$$

where α is any number.

The considered examples illustrate the substitution method.

To solve a system of two equations of the first degree in two unknowns x and y, it is necessary:

to express one of the unknowns, say y, in terms of the other unknown x from one of the equations of the system;

to substitute the expression thus obtained for y into the other equation;

to solve the obtained linear equation in one unknown x. As we know, the latter equation may have:

(a) the only solution x_0;

(b) no solution;

(c) infinitely many solutions: any real number α may be its solution.

In Case (a), substituting the found number x_0 into the expression for y, we find y_0; the pair of numbers (x_0, y_0) will just be the only solution of the system.

In Case (b) the system has no solution.

In Case (c) the system has infinitely many solutions.

Using the substitution method, let us solve several systems.

Example 4. Solve by substitution:

$$\begin{cases} 4x - 5y - 3 = 0 \\ 7x - y + 6 = 0 \end{cases} \tag{7}$$

From the second equation of System (7) we express y in terms of x:

$$y = 7x + 6. \tag{8}$$

and substitute this expression into the first equation:

$$4x - 5(7x + 6) - 3 = 0. \tag{9}$$

We then solve linear equation (9):

$$4x - 35x - 30 - 3 = 0, \quad -31x - 33 = 0,$$

whence it is clear that Equation (9) has the only solution

$$x_0 = -\frac{33}{31}.$$

Substituting x_0 into (8), we find

$$y_0 = 7x_0 + 6 = 7 \cdot \left(-\frac{33}{31}\right) + 6 = \frac{-7 \cdot 33 + 6 \cdot 31}{31} = -\frac{45}{31}.$$

Hence, System (7) has the only solution

$$\left(-\frac{33}{31}, -\frac{45}{31}\right).$$

Example 5. Solve the system of equations

$$\begin{cases} x - 5y + 3 = 0, \\ 3x - 15y + 1 = 0. \end{cases} \tag{10}$$

From the first equation we find

$$x = 5y - 3.$$

Substituting into the second equation, we get

$$3(5y - 3) - 15y + 1 = 0, \tag{11}$$

or

$$-8 = 0.$$

Consequently, Equation (11) has no solution. Therefore System (10) has no solution, that is, it is said to be incompatible.

Example 6. Solve by substitution:

$$\begin{cases} 10x - 5y + 15 = 0, \\ 2x - y + 3 = 0. \end{cases} \tag{12}$$

From the second equation we find

$$y = 2x + 3.$$

Substituting into the first, we get

$$10x - 5(2x + 3) + 15 = 0.$$

This linear equation is actually an identity

$$0 = 0,$$

therefore its solution is any number α.

Hence, System (12) has infinitely many solutions represented by pairs of numbers $x_0 = \alpha$, $y_0 = 2\alpha + 3$, where α is any real number.

SECTION TEST

QUESTIONS

1. What is the essence of the substitution method?

EXERCISES

1. Solve by substitution:

(a) $\begin{cases} 2x-3y+7=0, \\ 3x+4y-1=0; \end{cases}$ (b) $\begin{cases} 3x+4y-5=0, \\ 6x+8y+11=0; \end{cases}$ (c) $\begin{cases} x-5y+1=0, \\ 3x-15y+3=0. \end{cases}$

Sec. 14.4. The Addition Method

Example 1. Solve the system of equations

$$\begin{cases} 2x+y-3=0, \\ 3x-y-2=0. \end{cases} \tag{1}$$

Let the pair of numbers (x, y) be a solution of System (1). Substituting them into the equations of System (1), we get true numerical equalities. We see that the coefficients of y in these equalities differ only in sign. Adding these equalities, we get the equality

$$5x - 5 = 0,$$

with y eliminated, whence $x = 1$.

Substituting x into one of the equalities (1), for definiteness into the second, we get

$$3 - y - 2 = 0.$$

whence

$$y = 1.$$

Thus, we have shown that if System (1) has a solution, then it may be only the pair of numbers (1, 1). Substituting these numbers into the equations of System (1), we make sure that they satisfy these equations.

Consequently, System (1) has the only solution (1, 1).

Example 2. Solve the system of equations

$$\begin{cases} 3x-7y+1=0, \\ 6x-14y+3=0. \end{cases} \tag{2}$$

Let the pair of numbers (x, y) be a solution of System (2). Substituting them into the equations of the systems, we turn them into true equalities. We then transform these numerical equalities so that to

obtain equal coefficients of x in both equalities. To this end, we multiply the first equality by 2 to get the following equalities:

$$\begin{cases} 6x-14y+2=0, \\ 6x-14y+3=0. \end{cases}$$

Subtracting the second equality from the first, we get

$$-1 = 0.$$

The obtained contradiction means that System (2) has no solution.

Example 3. Solve the system of equations

$$\begin{cases} 2x-4y+2=0, \\ 3x-6y+3=0. \end{cases} \tag{3}$$

We assume that the pair of numbers (x, y) is a solution of System (3), and substitute them into the equations of the system. By this we turn them into true numerical equalities. We then transform these equalities so that to get equal coefficients of x in both equalities. Dividing the first of them by 2 and the second by 3, we obtain the following equalities

$$\begin{cases} x-2y+1=0, \\ x-2y+1=0. \end{cases}$$

Subtracting the second equality from the first, we get

$$0 = 0.$$

This equality indicates that y may be any number which will be denoted by α : $y = \alpha$.

Substituting α for y into either of the equalities of (3), for definiteness in the first, we get

$$2x - 4\alpha + 2 = 0,$$

whence

$$x = 2\alpha - 1.$$

It is easy to see that for any α the pair of numbers $x = 2\alpha - 1$, $y = \alpha$ satisfies System (3).

Consequently, System (3) has infinitely many solutions, all of them have the form

$$(2\alpha - 1, \alpha),$$

where α is any number.

From the aforegoing examples there follows the addition method.

To solve a system of two first-degree equations in two unknowns it is necessary to do the following:

multiplying by numbers, make the coefficients, say of x, equal in both equations,

subtracting the second equation from the first, get a linear equation in one unknown y; the latter equation may have:

(a) the only solution y_0;

(b) no solution;

(c) as a solution any real number α.

In Case (a), substituting the found number y_0 into any equation of the system, we find x_0; the pair of numbers (x_0, y_0) will be the only solution of the system;

in Case (b) the system has no solution;

in case (c) the system has infinitely many solutions.

Let us solve several systems applying the addition method.

Example 4. Solve by addition:

$$\begin{cases} 3x+4y+5=0, \\ 2x+3y+1=0. \end{cases} \tag{4}$$

Multiplying the first equation of this system by 2, and the second by 3 and subtracting then the second equation from the first we get the linear equation in one unknown y:

$$-y+7=0,$$

whence $y = 7$. Substituting 7 for y into the first equation of System (4), we get

$$3x+28+5=0,$$

whence $x = -11$.

Consequently, System (4) has the only solution $(-11, 7)$.

Example 5. Solve the system of equations

$$\begin{cases} 2x-5y+7=0, \\ 4x-10y+6=0. \end{cases} \tag{5}$$

Dividing the second equation by 2 and subtracting then the second equation from the first, we get the linear equation in one unknown:

$$0 \cdot y + 4 = 0.$$

This equation has no solution, consequently, System (5) also has no solution.

Example 6. Solve the system of equations

$$\begin{cases} 4x-2y+2=0, \\ 6x-3y+3=0. \end{cases} \tag{6}$$

Dividing the first equation by 2 and the second by 3, we get the system of equations

$$\begin{cases} 2x-y+1=0, \\ 2x-y+1=0, \end{cases} \tag{7}$$

which is equivalent to System (6). Subtracting the second equation from the first, we get the identity

$$0 = 0,$$

which shows that System (6) has infinitely many solutions.

Setting $x = \alpha$ where α is any number, and substituting α for x into any equation of System (6), we find

$$y = 2\alpha + 1.$$

Consequently, all solutions of System (6) have the form

$$(\alpha,\ 2\alpha + 1),$$

where α is any number.

Remark. It is obvious that the solutions of System (7) coincide with the solutions of one equation of this system, therefore on getting System (7), we may write out the solutions of this equation: these will just be the solutions of the system.

SECTION TEST

QUESTIONS

1. What is the essence of the addition (subtraction) method?

EXERCISES

1. Solve by addition or subtraction:

(a) $\begin{cases} 7x-y+1=0, \\ 2x+y-3=0; \end{cases}$ (b) $\begin{cases} 3x-4y-2=0, \\ 6x-3y-4=0; \end{cases}$ (c) $\begin{cases} 9x-3y+6=0, \\ 3x-y+2=0. \end{cases}$

Sec. 14.5. Equivalence of Equations and Systems of Equations

Hereafter, we shall come across equations whose both sides are either zeros or polynomials of not higher than first degree with respect to x and y. Such equations will be also called *linear equations in two unknowns x and y.*

Here are examples of such linear equations:

$$2x - 3y + 1 = 0; \quad 5x - 4y = 3x - 1;$$
$$2x - 3y = 5; \quad 3x - y + 1 = 3x + y - 1;$$
$$2x - 5y + 2 = 2x - 5y + 1; \quad 6x - 7y + 9 = 6x - 7y - 9.$$

The terms of the polynomials found in both sides of a linear equation are called the *terms* of this equation.

Recall that two equations are called equivalent if any solution of the first equation is a solution of the second, and any solution of the second equation is a solution of the first.

If an equation is multiplied (or divided) by a number different from zero, then an equivalent equation is obtained.

For instance, the equations

$$2x - 3y + 1 = 0$$

and

$$4x - 6y + 2 = 0$$

are equivalent.

If in a linear equation we collect like terms or transpose the terms (with reverse signs) from its left-hand or right-hand side to the other side, then we get an equation equivalent to the original equation. For instance, the equations

$$2x - y + 3x - 4 = x - 1$$

and

$$4x - y - 3 = 0$$

are equivalent.

In general, if in any linear equation all the terms are transposed to the left-hand side, then it turns out to be equivalent to the linear equation

$$ax + by + c = 0,$$

where a, b, and c are some numbers.

Two systems of equations are called *equivalent* if any solution of the first system is a solution for the second system and any solution of the second system is a solution for the first system.

Obviously, if one of the equations of a system is replaced by an equivalent equation, then the obtained system will be equivalent to the original system.

For instance, the system

$$\begin{cases} 2x+3y-1=0, \\ 4x+7y-5=0 \end{cases}$$

is equivalent to the system

$$\begin{cases} y=-\frac{2}{3}x+\frac{1}{3}, \\ 4x+7y-5=0. \end{cases}$$

SECTION TEST

QUESTIONS

1. What two equations are called equivalent?
2. What statements can be made about the equivalence of linear equations?
3. What two systems of equations are called equivalent?
4. What statements can be made about the equivalence of systems of equations?

EXERCISES

1. Find out which of the two given systems of equations are equivalent:

(a) $\begin{cases} x-2y+3=0, \\ 2x+y-4=0 \end{cases}$ and $\begin{cases} 3x-4y+5=0, \\ 7x-2y-3=0; \end{cases}$

(b) $\begin{cases} 3x-5y+1=0, \\ 4x+5y+1=0 \end{cases}$ and $\begin{cases} x+y+1=0, \\ 2x-4y+3=0. \end{cases}$

Sec. 14.6. Worked Problems

The solution of some problems is reduced to solving systems of two linear equations in two unknowns. As an example, we take the following old problem.

Problem 1. There met two shepherds, Ivan and Pyotr. Ivan said to Pyotr: "Just give me one sheep, then I will have sheep twice as many as you have!" Pyotr replied: "No, better you would give me one sheep, then we will have equal amount of sheep!" How many sheep had each shepherd?

Solution. Let Ivan have x sheep, and Pyotr y sheep. If Pyotr gave Ivan one sheep, then Pyotr would have $(y-1)$ sheep, and Ivan $(x+1)$ sheep. But then Ivan would have twice as many sheep as Pyotr did. Consequently,

$$x+1=2(y-1)$$

If Ivan gave Pyotr one sheep, then Ivan would have $(x-1)$ sheep, and Pyotr $(y+1)$ sheep. But then they would have equal amount of sheep. Consequently,

$$x-1=y+1.$$

Thus, we have obtained the system of two linear equations in two unknowns

$$\begin{cases} x+1=2(y-1), \\ x-1=y+1. \end{cases} \tag{1}$$

Let us solve this system using the substitution method.

From the first equation we have

$$x=2y-3 \tag{2}$$

Substituting $2y-3$ for x into the second equation, we get

$$2y-3-1=y+1.$$

which has the only solution $y_0=5$. Substituting $y_0=5$ into (2), we find $x_0=7$. Consequently, System (1) has the only solution: $x_0=7$, $y_0=5$, that is, Ivan had 7 sheep, and Pyotr 5 sheep.

Problem 2. A cyclist started from point A towards point B, and in fifteen minutes he was followed by a car. The car overtook the cyclist exactly midway between A and B. When the car reached the

point B, it remained for the cyclist to cover the last third of his way. How much time did it take the cyclist and the car to cover the path from A to B if it is known that both the cyclist and the car moved with a constant speed?

Solution. Let x denote the time (in minutes) during which the cyclist covered the path from A to B, and let y denote the time (in minutes) during which the car covered the same way. It took the cyclist $\frac{1}{2}x$ minutes to ride half the way from A to B, and it took the car $\frac{1}{2}y$ minutes for the same purpose. By the hypothesis, they were at the middle of the path at the same time, although the car left the point A fifteen minutes later. Hence,

$$\frac{1}{2}y+15=\frac{1}{2}x.$$

By the instant of the arrival of the car at the point B the cyclist had been riding already $(y + 15)$ minutes and covered during this time two thirds the path from A to B, that is, it took him $\frac{2}{3}x$ minutes to cover this path. Consequently,

$$y+15=\frac{2}{3}x.$$

We have obtained a system of two linear equations in two unknowns:

$$\begin{cases} \frac{1}{2}y+15=\frac{1}{2}x, \\ y+15=\frac{2}{3}x. \end{cases} \tag{3}$$

Multiplying the first equation by 2 and then subtracting the second equation from it, we get the linear equation in one unknown x: $15 = \frac{1}{3}x$, whence we find its only solution $x = 45$. And then from the second equation of (3) we find $y = 15$.

Thus, it took the cyclist 45 minutes and the car 15 minutes to cover the path from A to B.

SECTION TEST

EXERCISES

1. It is known that the sum of the ages of brother and sister is equal to 15 and the difference to 3. How old are brother and sister?

2. Two pieces of the same cloth cost together 91 roubles. On selling from the first piece as many metres of cloth as the second piece originally contained, and from the second piece half the amount of cloth contained originally in the first piece, the remnant of the first piece turned out to exceed the remnant of the second piece by 10 metres. How many metres of cloth did each piece contain if 1 metre costs 1 rouble and 40 copecks?

Chapter 15

LINEAR FUNCTIONS

Sec. 15.1. The Graph of Directly Proportional Dependence

Example 1. Let us be given in the plane the rectangular coordinate system Oxy and a straight line passing through the origin O (0, 0) and the point B (1, 3) (Fig. 16).

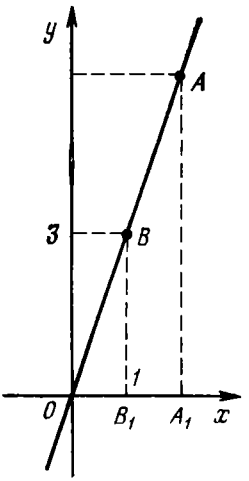


Fig. 16

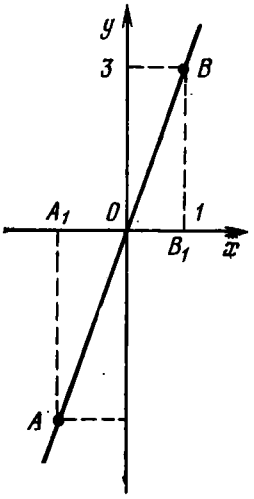


Fig. 17

Let A (x, y) be an arbitrary point of this line. Marked in Fig. 16 is just such a point having a positive abscissa x. The points A_1 and B_1 are the respective projections of the points A and B on the axis of abscissas. It is obvious that

$$|OB_1| = 1; \quad |BB_1| = 3; \quad |OA_1| = x; \quad |A_1A| = y.$$

Marked on this line is a point A having a negative abscissa x (Fig. 17). As before, A_1 and B_1 are projections of the points A and B on the axis of abscissas, but now

$$|OB_1| = 1; \quad |BB_1| = 3; \quad |OA_1| = -x; \quad |A_1A| = -y.$$

The right triangles OB_1B and OA_1A have congruent angles: $\angle BOB_1 = \angle AOA_1$ (in case of Fig. 16 these angles coincide, while in case of Fig. 17 they are equal as vertical angles). But (as is known from geometry) the corresponding legs of these triangles are proportional, that is,

$$\frac{|AA_1|}{|OA_1|} = \frac{|B_1B|}{|OB_1|}$$

or

$$\frac{y}{x}=\frac{3}{1},$$

whence

$$y = 3x. \qquad (1)$$

Obviously, the coordinates of the points B (1, 3) and O (0, 0) satisfy Equality (1).

Thus, if the point A (x, y) lies on the straight line under consideration, then

$$y = 3x.$$

Let us now prove the converse, namely: if a point A (x, y) is such that $y = 3x$, then it lies on the straight line in question.

Consider a point A (x, y), where $y = 3x$, such that x is positive, and the point B (1, 3) (Fig. 18). Let A_1 and B_1 be the respective projections of the points A and B on the axis of abscissas. It is obvious that

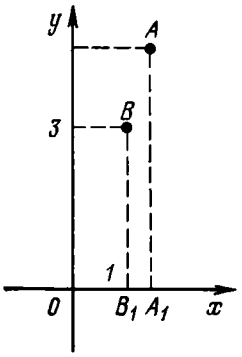


Fig. 18

$$|OA_1| = x; \quad |AA_1| = y;$$
$$|OB_1| = 1; \quad |BB_1| = 3.$$

Taking into account that $y = 3x$, we get

$$\frac{|AA_1|}{|OA_1|}=\frac{y}{x}=\frac{3x}{x}=\frac{3}{1}.$$

Since $\frac{|BB_1|}{|OB_1|}=\frac{3}{1}$, we obtain

$$\frac{|AA_1|}{|OA_1|}=\frac{|BB_1|}{|OB_1|}.$$

Thus, the right triangles OB_1B and OA_1A have proportional legs:

$$\frac{|AA_1|}{|OA_1|}=\frac{|BB_1|}{|OB_1|}.$$

but then (and this is also proved in geometry) the angles A_1OA and B_1OB are congruent in these triangles. In other words, then the points O, B, and A lie on one straight line (they are collinear). It is possible to show that any point A (x, y), where $y = 3x$ and x is negative, also lies on the same line.

Thus,

if a point A (x, y) lies on the straight line passing through the origin and the point B (1, 3), then $y = 3x$;

and if a point A (x, y) is such that $y = 3x$, then it lies on the straight line passing through the origin and the point B (1, 3).

The straight line passing through the origin and the point B (1, 3) is said to be the graph of the function $y = 3x$.

This straight line is also said to have the equation $y = 3x$.

Example 2. Let us specify in the coordinate plane Oxy a straight line passing through the origin and the point B (1, -2) and let us prove that this line has the equation $y = -2x$ (Fig. 19).

Indeed, let A (x, y) be an arbitrary point on our straight line. As before, A_1 and B_1 are the respective projections of the points A and B on the x-axis. It is clear that

$$|OB_1| = 1; \quad |BB_1| = 2.$$

If the point A is located to the right of the origin (that is, if $x > 0$), then

$$|OA_1| = x; \quad |AA_1| = -y$$

And if the point A_1 is situated to the left of the origin (that is, if $x < 0$), then

$$|OA_1| = -x; \quad |AA_1| = y$$

Again, the right triangles OB_1B and OA_1A have congruent angles: $\angle BOB_1 = \angle AOA_1$. But then (as in Example 1)

$$\frac{|AA_1|}{|OA_1|} = \frac{|BB_1|}{|OB_1|}.$$

Therefore in each of the considered cases (both for $x > 0$ and $x < 0$)

$$-\frac{y}{x} = \frac{2}{1},$$

whence

$$y = -2x.$$

Obviously, for $x = 0$, $y = 0$ this equality is also true.

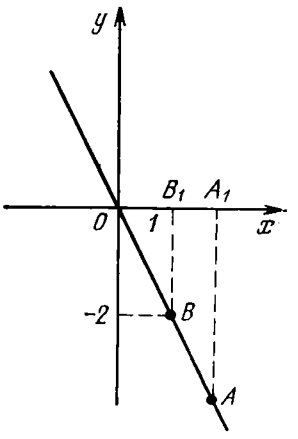


Fig. 19

Thus, if the point A (x, y) lies on the straight line passing through the origin and the point B (1, -2), then $y = -2x$. Reasoning exactly in the same way as in Example 1, it is possible to show that if the point A (x, y) is such that $y = -2x$, then it lies on the straight line passing through the origin and the point B (1, -2).

Consequently, the straight line passing through the origin and the point B (1, -2) has the equation $y = -2x$.

The examples considered in this and the preceding sections may be generalized. Namely, it is possible to consider in the coordinate plane Oxy a straight line passing through the origin and the point

$B(1, k)$, where k is an arbitrarily given number. Reasoning as above, we can show that such a line has the equation

$$y = kx.$$

In other words, it is possible to show that the graph of the function

$$y = kx$$

is a straight line passing through the origin and the point $B(1, k)$.

Note that if this straight line is the axis of abscissas, then any of its points has the ordinate $y = 0$ irrespective of its abscissa.

Thus, the equality

$$y = 0$$

should be regarded as the equation of the x-axis. It is obtained from the equation $y = kx$ for $k = 0$.

Thus, the graph of the function

$$y = kx,$$

where k is some number (positive, negative, or zero), is a straight line passing through the origin and the point $B(1, k)$.

Note that the function $y = kx$, where $k \neq 0$, expresses a *directly proportional dependence between x and y*. Indeed, let y_1, y_2 be values of this function corresponding to the values x_1, x_2 of the independent variable, that is, let

$$y_1 = kx_1, \quad y_2 = kx_2.$$

Then for $k \neq 0$ and $x_1 \neq 0$ the following proportion holds true:

$$\frac{y_2}{y_1} = \frac{kx_2}{kx_1} = \frac{x_2}{x_1}.$$

Also note that instead of saying "given a straight line whose equation is $y = kx$", we often say "given a straight line $y = kx$".

The number k in the equation of the straight line $y = kx$ is called the *slope* of this line.

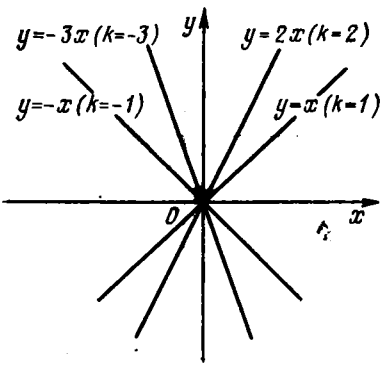


Fig. 20

Figure 20 represents different straight lines passing through the origin with their slopes indicated in parentheses.

Note that if the slope k of a straight line is positive ($k > 0$), then it makes an acute angle with the positive direction of the x-axis, and if the slope of a straight line is negative ($k < 0$), then it forms an obtuse angle with the positive direction of the x-axis (Fig. 20).

SECTION TEST

QUESTIONS

1. What is the graph of the function:
(a) $y = 3x$; (b) $y = -2x$; (c) $y = x$; (d) $y = 0$?
2. How do we write the equation of a straight line passing through the origin and the point $(1, k)$, where k is a given number?
3. What is called the slope of the straight line $y = kx$?

EXERCISES

1. Write the equation of the straight line passing through the origin and the point:

(a) $B\left(1, \frac{1}{2}\right)$; (b) $B\,(1, -3)$; (c) $B\left(1, -\frac{1}{2}\right)$; (d) $B\,(1, 5)$.

2. On the coordinate plane Oxy, graph the following functions:

(a) $y = \frac{1}{3}x$; (b) $y = \frac{3}{4}x$; (c) $y = \frac{5}{2}x$; (d) $y = -\frac{7}{3}x$; (e) $y = 0$;

(f) $y = -x$.

Sec. 15.2. A Linear Function

A function of the form

$$y = kx + b \tag{1}$$

where k and b are given numbers, is called a *linear function.*

If $b = 0$, then we get the function $y = kx$ which was dealt with in the preceding section.

Let us give a number of examples of linear functions:

$$y = 2x + 3; \quad y = 2x - 3; \quad y = x + 5;$$
$$y = -x + 5; \quad y = -\frac{1}{2}x + 2; \quad y = -2x + 3.$$

To draw the graph of the function

$$y = 2x + 3 \tag{2}$$

in the rectangular coordinate system Oxy (Fig. 21), let us first represent the graph of the function $y = 2x$. This is a straight line passing through the points $O\,(0, 0)$ and $B\,(1, 2)$.

If this straight line is displaced parallel to itself upwards by 3 scale units, then the graph of the function $y = 2x + 3$ is obtained, since if A is an arbitrary point of the graph of the function $y = 2x$ and C is a point of the graph of the function $y = 2x + 3$ having the same abscissa x, then, obviously, the ordinate of the point C is greater than the ordinate of the point A by 3 units.

Thus, the straight line $y = 2x + 3$ is parallel to the line $y = 2x$. Besides, the line $y = 2x + 3$ intersects the y-axis at the point D (0, 3), which may be verified by setting $x = 0$ in the equation $y = 2x + 3$.

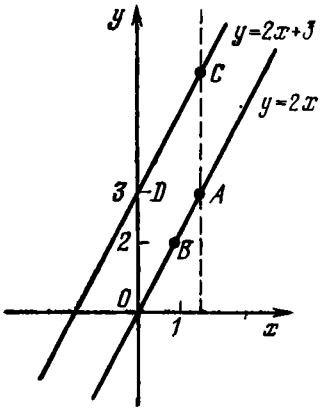


Fig. 21

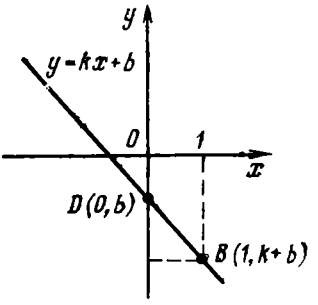


Fig. 22

The above reasoning concerning the function $y = 2x + 3$ may be, by analogy, generalized for any linear function $y = kx + b$, where k and b are any given numbers.

Thus, the *graph of the linear function*

$$y = kx + b$$

is a straight line intersecting the axis of ordinates at the point D (0, b) and is parallel to the line $y = kx$.

It is easily seen that the point B (1, $k + b$) lies on the line

$$y = kx + b.$$

Therefore we may say so: the graph of the linear function (Fig. 22)

$$y = kx + b$$

is a straight line passing through two points

$$D\ (0,\ b) \quad \text{and} \quad B\ (1,\ k + b).$$

The coefficient k in the equation

$$y = kx + b \tag{3}$$

is called the *slope* of this line

Note that two straight lines

$$y = kx + b \quad \text{and} \quad y_1 = k_1x + b_1,$$

having equal slopes k and k_1 ($k = k_1$) and different numbers b and b_1 ($b \neq b_1$), are parallel.

If $k = 0$, then we get the function

$$y = b, \qquad (4)$$

which is independent of x. This function expresses the following rule (law): to every value of x there corresponds one and the same number $y = b$. Such function is called *constant*. Its graph is a straight line parallel to the x-axis and intersecting the y-axis at the point $(0, b)$ (Fig. 23). Its slope is equal to zero.

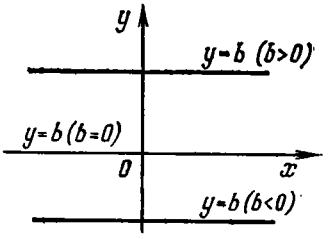


Fig. 23

SECTION TEST

QUESTIONS

1. What function is called linear?
2. What is the graph of a linear function?

EXERCISES

1. Draw the graphs of the following linear functions:
(a) $y = 3x + 2$; (b) $y = 3x - 2$; (c) $y = x + 3$;
(d) $y = -x + 2$; (e) $y = \frac{1}{2}x - 4$; (f) $y = -3x + 5$.

2. Determine the coordinates of the points at which the straight lines listed in Exercise 1 intersect the x-axis.

3. Which of the straight lines from Exercise 1 are parallel?

4. Find the equation of the straight line passing through the point (0, 3) parallel to the line:

(a) $y = 2x$; (b) $y = 2x - 1$; (c) $y = \frac{3}{2}x - 1$; (d) $y = -x + 2$.

Sec. 15.3. Uniform Motion

Example 1. Let us be given a coordinate axis s with the origin O and unit segment of length 1 (centimetre).

Suppose a point started at the instant of time $t = 0$ from the point on the axis s having the coordinate 3 and moves uniformly in the positive direction of this axis with a velocity of 2 cm/s. The coordinate s (cm) of this point is, obviously, a function of time t (s) expressed by the formula

$$s = 3 + 2t.$$

The given function is considered for positive values of t and $t = 0$, therefore it is said to be defined for nonnegative values of t ($t \geqslant 0$).

Let us introduce the rectangular coordinate system Ots and in it represent the graph of the function

$$s = 3 + 2t \quad (t \geqslant 0), \tag{1}$$

or, as we say, the graph of point motion. This is a ray emanating from the point C (0, 3) parallel to the straight line

$$s = 2t$$

(see Fig. 24).

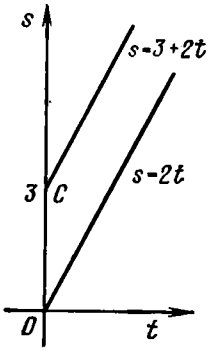


Fig. 24

Using this graph, it is possible to study the motion under consideration.

For instance, to find out where our point is located at a given instant of time t, it is necessary to lay off on the axis of abscissas to the right of O a line segment OA_1 of length t ($| OA_1 | = t$) and to erect at A_1 a perpendicular to the axis of abscissas to intersect the graph of motion at some point A. The number s, equal to the length of the line segment

$$s = | AA_1 |,$$

is the coordinate (on the axis s) of our moving point at the instant of time t.

Example 2. Let us be given a coordinate axis s with the origin O and unit segment of length 1 (centimetre).

Let a point move along the axis s. In this motion, its coordinate s is a linear function of time expressed by the formula

$$s = 4t + 2. \tag{2}$$

Setting $t = 0$ in this formula, we get $s = 2$. This means that our moving point had the coordinate $s = 2$ at the instant $t = 0$.

Let us take two arbitrary instants t_1 and t_2, where $t_1 < t_2$. At the instant $t = t_1$ our point has the coordinate $s = s_1$ computed by the formula

$$s_1 = 4t_1 + 2.$$

At the instant $t = t_2$ our point has the coordinate $s = s_2$ computed by the formula

$$s_2 = 4t_2 + 2.$$

The time interval between the instants t_1 and t_2 is equal to $t_2 - t_1$. The path covered by the point during this time interval is, obviously, equal to

$$s_2 - s_1 = (4t_2 + 2) - (4t_1 + 2) = 4(t_2 - t_1).$$

Hence, the velocity of our point is equal to

$$v=\frac{s_2-s_1}{t_2-t_1}=\frac{4(t_2-t_1)}{t_2-t_1}=4.$$

Thus, if the point moves along the s-axis according to the rule (law) expressed by Formula (2), then it moves uniformly with a velocity of 4 cm/s, and at the instant $t = 0$ it was at the point $s = 2$.

SECTION TEST

EXERCISES

1. Write the equation of motion of a point moving along the s-axis:
(a) with a velocity of 4 cm/s if at the instant $t = 0$ it has the coordinate $s = 5$;
(b) with a velocity of 6 cm/s if at the instant $t = 0$ it has the coordinate $s = 2$.
Draw the graphs of motion.
2. Given the equation of motion of a point moving along the s-axis:

(a) $s=2t-7$; (b) $s=\frac{1}{2}t+3$; (c) $s=3t$.

Determine the coordinate of the point at the instants $t = 0$ and $t = 3$.
Also determine the velocity of the point.
Draw the graph of motion.

Sec. 15.4. Graphical Method of Solving a System of Linear Equations

Example 1. Solve graphically the system of equations

$$\begin{cases} 3x-y+2=0, \\ -x+2y+1=0. \end{cases} \tag{1}$$

Solving each of these equations with respect to y, we get the system

$$\begin{cases} y=3x+2, \\ y=\frac{1}{2}x-\frac{1}{2}, \end{cases} \tag{2}$$

which is equivalent to System (1).

Let us introduce the rectangular coordinate system Oxy. We know that the equation $y = 3x + 2$ is the equation of the straight line passing through the points A_1 (1, 5) and B_1 (0, 2) and $y=\frac{1}{2}x-\frac{1}{2}$ is the equation of the straight line passing through the points A_2 (1, 0) and $B_2\left(0,-\frac{1}{2}\right)$. Let us draw these lines in the system Oxy (Fig. 25). As we see, they intersect at the point M (—1, —1). Its coordinates $x = -1$, $y = -1$ are just the only solution of System (2), but then also of System (1).

Note that, judging by the form of Equations (2), we can say beforehand that the considered lines intersect at one point. In fact, the slopes of these lines are different $\left(3 \neq \frac{1}{2}\right)$, which indicates that the lines are not parallel and, consequently, intersect at one point.

Example 2. Solve graphically the system of equations

$$\begin{cases} x-y+1=0, \\ x-y+2=0. \end{cases} \tag{3}$$

Solving each of these equations with respect to y, we get the system of equations equivalent to System (3):

$$\begin{cases} y=x+1, \\ y=x+2. \end{cases} \tag{4}$$

These are equations of parallel lines since they have equal slopes. These lines do not coincide since they intersect the y-axis at different

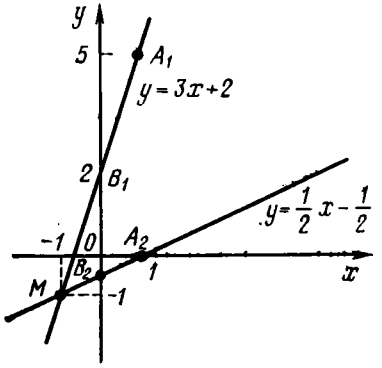


Fig. 25

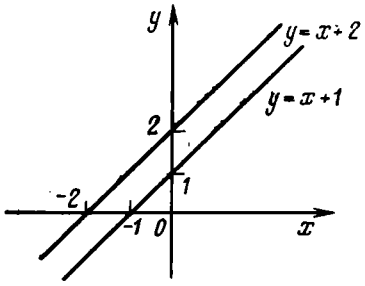


Fig. 26

points: the first at the point (0, 1), the second at the point (0, 2) (Fig. 26). Thus, these lines do not intersect, hence, System (4) and, consequently, System (3) have no solutions.

Example 3. Solve graphically the system of equations

$$\begin{cases} 2x+2y-1=0, \\ -4x-4y+2=0. \end{cases} \tag{5}$$

Solving each of these equations with respect to y, we get the system

$$\begin{cases} y=-x+\frac{1}{2}, \\ y=-x+\frac{1}{2}, \end{cases} \tag{6}$$

which is equivalent to System (5).

System (6) consists of two equal equations and, consequently, defines one and the same straight line $y = -x + \frac{1}{2}$.

This shows that all the solutions of System (5) form the totality of coordinate pairs (x, y) of points belonging to this line.

SECTION TEST

EXERCISES

1. Solve graphically the following systems of equations:

(a) $\begin{cases} x-y+1=0, \\ 2x+y-1=0; \end{cases}$ (b) $\begin{cases} 7x-y-3=0, \\ 14x-2y+5=0; \end{cases}$ (c) $\begin{cases} 3x+y-1=0, \\ 6x+2y-2=0. \end{cases}$

Sec. 15.5. Investigating a System of Two First-degree Equations in Two Unknowns

Let us now consider the general system of equations

$$\begin{cases} a_1x+b_1y+c_1=0, \\ a_2x+b_2y+c_2=0, \end{cases} \tag{1}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are arbitrary numbers different from zero. Solving these equations with respect to y, we get the system

$$\begin{cases} y=-\frac{a_1}{b_1}x-\frac{c_1}{b_1}, \\ y=-\frac{a_2}{b_2}x-\frac{c_2}{b_2}, \end{cases} \tag{2}$$

which is equivalent to (1).

The first equation of System (2) is the equation of the straight line (in the Cartesian coordinate system Oxy) with the slope $-\frac{a_1}{b_1}$. The second equation of System (2) is the equation of the straight line with the slope $-\frac{a_2}{b_2}$.

Three cases may occur.

Case 1. The coefficients of x and y of the equations of System (1) are not proportional, that is,

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

Then the slopes of our lines are different:

$$-\frac{a_1}{b_1} \neq -\frac{a_2}{b_2},$$

and the lines intersect only at one point. Consequently, System (1) has the only solution.

Case 2. The coefficients of x and y of the equations of System (1) are proportional, but they are not proportional to the constant terms, that is,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}.$$

Then the slopes of our lines are equal to each other:

$$-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$$

but

$$-\frac{c_1}{b_1}\neq-\frac{c_2}{b_2}$$

and the lines are parallel and do not coincide. Consequently, System (1) has no solutions at all.

Case 3. The numbers a_1, b_1, c_1 are respectively proportional to the numbers a_2, b_2, c_2, that is,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

Then, obviously,

$$-\frac{a_1}{b_1}=-\frac{a_2}{b_2},\quad -\frac{c_1}{b_1}=-\frac{c_2}{b_2},$$

and, consequently, both equations of System (2) define the same line. The coordinates of points of this line are just all possible solutions of System (1).

Remark. We shall assume that only the numbers a_1, b_1, a_2, b_2 are different from zero.

Let $c_1 = c_2 = 0$, then if the numbers a_1, b_1 are proportional to the numbers a_2, b_2, the lines under consideration coincide. And if these numbers are not proportional, then the lines intersect at the point (0, 0).

For $c_1 = 0$, $c_2 \neq 0$, if the numbers a_1, b_1 are proportional to the numbers a_2, b_2, then the lines are distinct and parallel; and if there is no proportionality of numbers, then the lines intersect.

Chapter 16

SYSTEMS OF RATIONAL EQUATIONS

We know how to solve systems of two linear equations in two unknowns.

In the present chapter, by considering a number of examples, we shall learn how to solve systems of n linear equations in n unknowns. We shall also consider more complicated cases of systems

in n unknowns consisting of $n - 1$ linear equations and one second-degree equation.

Some problems which we consider in detail below lead to still more complicated systems of equations, but their solution is reduced to solving the systems indicated above.

All systems to be considered below may be called systems of rational equations. An equation is called *rational* if both of its members (sides) are rational expressions with respect to the unknowns.

Sec. 16.1. Systems of First-degree Equations

In Chapter 9 we already studied systems of two first-degree equations in two unknowns and showed that such systems can be solved by the substitution method. Below, we give examples of solving a system of three first-degree equations in three unknowns and a system of four first-degree equations in four unknowns and show that these systems can also be solved by the substitution method.

Example 1. Solve the system of equations

$$\begin{cases} 2x - 3y + z - 1 = 0, \\ 3x - 4y - z + 2 = 0, \\ x - y + z = 0. \end{cases} \tag{1}$$

Let us show how to solve this system using the substitution method.

From the third equation of System (1) we express x in terms of y and z:

$$x = y - z, \tag{2}$$

and substitute $y - z$ for x into the first and second equations of System (1). We get the equations

$$\begin{aligned} 2(y - z) - 3y + z - 1 &= 0, \\ 3(y - z) - 4y - z + 2 &= 0, \end{aligned}$$

which, on collecting like terms, will be written in the form

$$\begin{aligned} -y - z - 1 &= 0, \\ -y - 4z + 2 &= 0. \end{aligned} \tag{3}$$

Thus, the substitution method has reduced the solution of the system of three first-degree equations in three unknowns x, y, and z to solving a system of two first-degree equations in two unknowns y and z which we know how to solve. Finding the solution $y = y_0$, $z = z_0$ of System (3), we substitute the numbers y_0 and z_0 for y and z into the right-hand side of (2). We get

$$x_0 = y_0 - z_0.$$

The numbers

$$x = x_0, \quad y = y_0, \quad z = z_0$$

just form the solution of System (1). (See Remark 1 below.)

Let us find these numbers. From the first equation of System (3) we express y in terms of z:

$$y = -z - 1 \tag{4}$$

and substitute this expression into the second equation of System (3) to get the equation in one unknown z

$$-(-z - 1) - 4z + 2 = 0,$$

or

$$-3z + 3 = 0,$$

whence we find the only solution $z_0 = 1$. Substituting z_0 into Expression (4), we find $y_0 = -2$; substituting z_0 and y_0 into Expression (2), we find $x_0 = -3$.

Thus, System (1) has the only solution: $x_0 = -3$, $y_0 = -2$, $z_0 = 1$.

In the general case, when solving a system of three first-degree equations in three unknowns x, y, and z, we may proceed as in Example 1. One of the unknowns, for instance z, is expressed in terms of the remaining unknowns entering one of the equations of the system. The expression thus obtained is substituted into two other equations. As a result, we get a system of two first-degree equations in two unknowns x and y.

If this system has a solution, then we find it (the numbers x_0 and y_0) and substitute them into the expression for z. And if this system has no solution, then the original system of three equations also has no solution.

But proceeding in such a way we may come across some simplifications. For instance, if one of the equations of a system contains only x, then, on solving it, we find at once the number x_0. Substituting then x_0 for x into the two remaining equations, we get two equations in two unknowns y and z. But it may happen that we have no chance to substitute x_0 since the remaining equations contain no x. Then it remains to solve them with respect to y and z.

The following example is just the case.

Example 2. Solve the system of equations

$$\begin{cases} x+y-1=0, \\ x+y-u+1=0. \\ 2x-3y-z-3=0, \\ x+y-z+u-4=0. \end{cases} \tag{5}$$

From the first equation we have

$$y = 1 - x. \tag{6}$$

Substituting $(1 - x)$ for y into the remaining equations of System (5), we get the system of three equations in three unknowns

$$\begin{cases} x+(1-x)-u+1=0, \\ 2x-3(1-x)-z-3=0, \\ x+(1-x)-z+u-4=0, \end{cases}$$

which, on collecting like terms, will be rewritten as follows:

$$\begin{cases} 2-u=0, \\ 5x-z-6=0, \\ -z+u-3=0. \end{cases} \tag{7}$$

From the first equation of System (7) we find $u = 2$. Substituting $u = 2$ into the third equation of System (7), we find $z = -1$. Substituting $z = -1$ into the second equation of System (7), we find $x = 1$. Finally, substituting $x = 1$ into (6), we find $y = 0$. Thus, System (5) has the only solution:

$$x = 1, \quad y = 0, \quad z = -1, \quad u = 2.$$

Remark. If everything was done correctly, the result needs no check. But some mistakes may slip in the computational work, and therefore it is recommended to substitute the obtained solution into the equations of the system and make sure that it actually satisfies these equations.

SECTION TEST

QUESTIONS

1. What is the essence of the substitution method for solving a system of three first-degree equations in three unknowns?

EXERCISES

1. Solve the following systems of equations:

(a) $\begin{cases} 2x-3y+z-1=0, \\ 3x-4y-z+2=0, \\ x+y+z=0; \end{cases}$ (b) $\begin{cases} 5x-7y=0, \\ x+z=12, \\ 3x-4y+z=6. \end{cases}$

Sec. 16.2. Solving Problems with the Aid of Systems of First-degree Equations

Problem. A pupil spent 28 roubles on buying a bag, a fountain pen, and a book. If the bag cost 5 times cheaper, the fountain pen 2 times cheaper, and the book 2.5 times cheaper, then the same purchase would cost 8 roubles. If, as compared with the actual price, the bag cost 3 times cheaper, the fountain pen 4 times cheaper, and

the book 2 times cheaper, then the same purchase would cost 10 roubles. What were the prices of the bag, the fountain pen, and the book?

Solution. Let the bag cost x roubles, the fountain pen y roubles, and the book z roubles. The first equation is set up from the conditions that the purchase costs 28 roubles:

$$x + y + z = 28.$$

The second equation is set up from the condition: if the bag cost $\frac{x}{5}$ roubles, the fountain pen $\frac{y}{2}$ roubles, and the book $\frac{z}{2.5}$ roubles, then the purchase would cost 8 roubles:

$$\frac{x}{5} + \frac{y}{2} + \frac{z}{2.5} = 8.$$

Finally, the third equation is set up from the condition: if the bag cost $\frac{x}{3}$ roubles, the fountain pen $\frac{y}{4}$ roubles, and the book $\frac{z}{2}$ roubles, then the purchase would cost 10 roubles:

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 10.$$

We have obtained the system of three first-degree equations in three unknowns:

$$\begin{cases} x + y + z = 28, \\ \frac{x}{5} + \frac{y}{2} + \frac{z}{2.5} = 8, \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 10. \end{cases} \tag{1}$$

From the first equation of System (1) we express z in terms of x and y:

$$z = 28 - x - y \tag{2}$$

and substitute the expression $(28 - x - y)$ for z into the second and third equations of System (1); as a result, we get the system of two equations in two unknowns x and y:

$$\begin{cases} \frac{x}{5} + \frac{y}{2} + \frac{2}{5}(28 - x - y) = 8, \\ \frac{x}{3} + \frac{y}{4} + \frac{1}{2}(28 - x - y) = 10, \end{cases}$$

which can be rewritten as follows:

$$\begin{cases} -2x + y + 32 = 0, \\ 48 - 2x - 3y = 0. \end{cases} \tag{3}$$

From the first equation of System (3) we express y in terms of x:

$$y = 2x - 32 \tag{4}$$

and substitute the expression $(2x - 32)$ for y into the second equation of System (3). As a result, we obtain the equation in one unknown x:

$$48 - 2x - 3(2x - 32) = 0,$$

which, on collecting like terms, will be written in the form

$$144 - 8x = 0. \tag{5}$$

Equation (5) has the only root $x_0 = 18$. Substituting x_0 into Expression (4), we find $y_0 = 4$. Substituting x_0 and y_0 into Expression (2) we find $z_0 = 6$.

Consequently, System (1) has the only solution: $x_0 = 18$, $y_0 = 4$, $z_0 = 6$, that is, the bag cost 18 roubles, the fountain pen 4 roubles, and the book 6 roubles.

SECTION TEST

EXERCISES

1. A point C is situated on the bank of a river at a distance of 12 km downstream. A fisherman set out in a boat for the point C from a point A situated higher than B. He reached C in four hours, and it took him six hours to complete the way back. Having mounted a motor on his boat and thus having increased three-fold the speed of the boat relative to the water, the fisherman covered the way from A to B for 45 minutes. Determine the distance from A to B, the rate of flow of the river assuming it to be constant, and the speed of the boat in stagnant water.

Sec. 16.3. Systems of Equations of First and Second Degrees

This section presents examples of solving a system of two equations in two unknowns one of which is an equation of the first degree, the other of the second, and systems of three equations in three unknowns two of which are equations of the first degree, the third being an equation of the second degree.

For solving such systems the substitution method is also applicable.

Example 1. Solve the system of equations

$$\begin{cases} x + 2y - 7 = 0, \\ x^2 + 2xy + y^2 + 3y - 4x - 31 = 0. \end{cases} \tag{1}$$

The first equation of this system is an equation of the first degree; from it we express x in terms of y:

$$x = 7 - 2y. \tag{2}$$

Substituting the expression $(7 - 2y)$ for x into the second equation of the system, we get

$$(7 - 2y)^2 + 2(7 - 2y)y + y^2 + 3y - 4(7 - 2y) - 31 = 0,$$

which, on collecting like terms, will be rewritten as follows:

$$y^2 - 3y - 10 = 0. \tag{3}$$

Equation (3) has two roots:

$$y_1 = -2 \quad \text{and} \quad y_2 = 5.$$

Substituting these numbers for y into Expression (2), we get

$$x_1 = 11 \quad \text{and} \quad x_2 = -3.$$

Thus, System (1) has two solutions:

$$x_1 = 11, \quad y_1 = -2; \quad x_2 = -3, \quad y_2 = 5,$$

and has no other solutions.

In such a way, it is possible to solve any system of two equations with the unknowns x and y in which one equation is of the first degree and the other of the second degree. We find x (or y) from the first-degree equation and substitute the result into the second-degree equation. We get a quadratic equation with respect to y (or x). If the quadratic equation has a real solution, then the system also has the corresponding pairs of solutions, if not, then the system has no solution.

Example 2. Solve the system of equations

$$\begin{cases} 3x - y + 9 = 0, \\ x^2 - y^2 + y - 5x - 32 = 0. \end{cases} \tag{4}$$

The first equation of System (4) is of the first degree and the second of the second degree. From the first equation we express y in terms of x:

$$y = 3x + 9.$$

Substituting the expression $(3x + 9)$ for y into the second equation of System (4), we get the equation

$$x^2 - (3x + 9)^2 + (3x + 9) - 5x - 32 = 0,$$

which, on collecting like terms will be rewritten as

$$-8x^2 - 56x - 104 = 0.$$

Dividing all the terms by (-8), we get the quadratic equation

$$x^2 + 7x + 13 = 0, \tag{5}$$

whose discriminant $D = b^2 - 4ac = 7^2 - 4 \cdot 1 \cdot 13 = -3 < 0$. Consequently, Equation (5) has no solution. Hence, System (4) also has no solution.

Example 3. Solve the system of equations

$$\begin{cases} x+y-z+1=0, \\ x-y-z+3=0, \\ x^2+2xy+y^2-xz+z^2+x-5=0. \end{cases} \tag{6}$$

From the first equation we express y in terms of x and z:

$$y = z - x - 1. \tag{7}$$

Substituting the expression $(z - x - 1)$ for y into the second and third equations of System (6), we get the system of two equations in two unknowns x and z:

$$\begin{cases} x-(z-x-1)-z+3=0, \\ x^2+2x(z-x-1)+(z-x-1)^2-xz+z^2+x-5=0, \end{cases}$$

which, on collecting like terms, will be written in the form

$$\begin{cases} 2x-2z+4=0, \\ 2z^2-xz+x-2z-4=0. \end{cases} \tag{8}$$

From the first equation of System (8) we express x in terms of z:

$$x = z - 2, \tag{9}$$

and, substituting the expression $(z - 2)$ for x into the second equation of System (8), we get the following equation in one unknown z:

$$2z^2 - z(z-2) + (z-2) - 2z - 4 = 0,$$

which, after simplification, will be written in the form

$$z^2 + z - 6 = 0. \tag{10}$$

Quadratic equation (10) has two roots:

$$z_{1,2} = \frac{-1 \pm \sqrt{(-1)^2 - 4 \cdot (-6) \cdot 1}}{2} = \frac{-1 \pm 5}{2},$$

that is,

$$z_1 = -3 \quad \text{and} \quad z_2 = 2.$$

Substituting these numbers into (9), we get: $x_1 = -5$, $x_2 = 0$. Finally, substituting first x_1 and z_1, and then x_2 and z_2 into (7), we find: $y_1 = 1$ and $y_2 = 1$. Hence, System (6) has two solutions:

$$\begin{aligned} x_1 &= -5, \quad y_1 = 1, \quad z_1 = -3; \\ x_2 &= 0, \quad y_2 = 1, \quad z_2 = 2, \end{aligned}$$

and has no other solutions.

SECTION TEST

QUESTIONS

1. How is it possible to solve a system of equations of the first and second degree?

EXERCISES

1. Solve the following systems of equations:

(a) $\begin{cases} x-y+3=0, \\ 2x^2-y^2+3x-y-9=0; \end{cases}$ (b) $\begin{cases} 2x+y+z=4, \\ x+4y+4z=-5, \\ xy+yz+zx=-9. \end{cases}$

Sec. 16.4. Solving Problems with the Aid of Systems of Equations of First and Second Degrees

Problem. The weight percent of acid in three solutions is such that the square of the weight percent of acid in the second solution is equal to the product of the weight percents of acid in the first and third solutions. If we mix the first, second, and third solutions in the weight ratios 2 : 3 : 4, then we get a solution containing 32% of acid, and if they are mixed in the weight ratios 3 : 2 : 1, then we get a solution containing 22% of acid. Determine the weight percent of acid contained in each solution.

Solution. Let the first solution contain x% of acid, the second y%, and the third z% of acid. By the first condition of the problem

$$y^2 = xz.$$

One gram of the first solution contains $\frac{x}{100}$ g of acid, one gram of the second solution $\frac{y}{100}$ g of acid, and one gram of the third solution $\frac{z}{100}$ g of acid. If we take 2 grams of the first solution, 3 grams of the second, and 4 grams of the third, then we get 9 grams of mixture containing

$$\left(2\cdot\frac{x}{100}+3\cdot\frac{y}{100}+4\cdot\frac{z}{100}\right) \text{ g}$$

of acid. According to the hypothesis, the obtained mixture contains 32% of acid, that is, 9 grams of the mixture contains $9\cdot\frac{32}{100}$ g of acid. From this condition we get the equation

$$2\cdot\frac{x}{100}+3\cdot\frac{y}{100}+4\cdot\frac{z}{100}=9\cdot\frac{32}{100}.$$

Reasoning in a similar way, we get one more equation

$$3 \cdot \frac{x}{100} + 2 \cdot \frac{y}{100} + 1 \cdot \frac{z}{100} = 6 \cdot \frac{22}{100}.$$

Thus, we have obtained a system of three equations in three unknowns. Let us rewrite it in the form

$$\begin{cases} y^2 = xz, \\ 2x + 3y + 4z = 288, \\ 3x + 2y + z = 132 \end{cases} \quad (1)$$

and solve it. From the third equation of System (1) we express z in terms of x and y:

$$z = 132 - 3x - 2y. \quad (2)$$

Substituting the expression $(132 - 3x - 2y)$ for z into the first and second equations of System (1), we get the system of two equations in two unknowns x and y:

$$\begin{cases} y^2 = x(132 - 3x - 2y), \\ 2x + 3y + 4(132 - 3x - 2y) = 288. \end{cases}$$

On transposing all terms of each equation to one side and collecting like terms, we get the system

$$\begin{cases} 3x^2 + y^2 + 2xy - 132x = 0, \\ 240 - 10x - 5y = 0. \end{cases} \quad (3)$$

From the second equation of this system we express y in terms of x:

$$y = 48 - 2x, \quad (4)$$

and substitute the expression $(48 - 2x)$ for y into the first equation of system (3). We get the equation in one unknown x

$$3x^2 + (48 - 2x)^2 + 2x(48 - 2x) - 132x = 0,$$

which, on collecting like terms, can be rewritten as

$$3x^2 - 228x + 2304 = 0.$$

Dividing this expression by 3, we get the quadratic equation

$$x^2 - 76x + 768 = 0. \quad (5)$$

The discriminant of the quadratic equation (5)

$$D = b^2 - 4ac = (-76)^2 - 4 \cdot 1 \cdot 768 = 52^2 > 0.$$

Hence, Equation (5) has two roots:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{76 \pm 52}{2},$$

that is, $x_1 = 64$, $x_2 = 12$.

Substituting x_1 and x_2 into Expression (4), we find

$$y_1 = -80, \quad y_2 = 24.$$

Substituting the pairs (x_1, y_1) and (x_2, y_2) into Expression (2), we get

$$z_1 = 100, \quad z_2 = 48.$$

Thus, System (1) has two solutions:

$$\begin{array}{lll} x_1 = 64, & y_1 = -80, & z_1 = 100; \\ x_2 = 12, & y_2 = 24, & z_2 = 48, \end{array}$$

but, by the supposition, y is the weight percent of acid in the second solution, and, hence, y cannot be a negative number. Consequently, only one solution satisfies the conditions of the problem:

$$x_2 = 12, \quad y_2 = 24, \quad z_2 = 48.$$

Thus, the first solution contained 12% of acid, the second 24%, the third 48%.

SECTION TEST

EXERCISES

1. The perimeter of a rectangle is equal to 16, its area to 15. Find the sides of this rectangle.
2. The square of the smaller of two natural numbers is equal to their sum, and the difference between these numbers is 15. Find these numbers.

Sec. 16.5. Solving Problems with the Aid of Systems of Rational Equations

Problem 1. A team of workers began digging a trench. After three days had elapsed a second team joined them to continue digging the same trench. It took both teams another eight days to complete the job. If, conversely, the first three days the job was carried out only by the second team, then another nine days were required for both teams to complete the job. How much time would it take each team to fulfil this job separately?

Solution. Suppose the first team can do the work during x days, and the second during y days. Then the first team carries out $\frac{1}{x}$ part of the total amount of work, per day, and the second $\frac{1}{y}$ part thereof.

In the first case the first team worked 11 days and, consequently, did $11 \cdot \frac{1}{x}$ parts of the entire job, while the second team worked

8 days and did $8 \cdot \frac{1}{y}$ parts of the total amount of work. Since they together fulfilled the job, we have

$$11 \cdot \frac{1}{x} + 8 \cdot \frac{1}{y} = 1.$$

In the second case the first team would work 9 days and would do $9 \cdot \frac{1}{x}$ parts of the work, and the second 12 days and would do $12 \cdot \frac{1}{y}$ parts. In this case, the combined efforts of the two teams would also complete the job, that is,

$$9 \cdot \frac{1}{x} + 12 \cdot \frac{1}{y} = 1.$$

We have obtained the system of two rational equations in two unknowns:

$$\begin{cases} 11 \cdot \frac{1}{x} + 8 \cdot \frac{1}{y} = 1, \\ 9 \cdot \frac{1}{x} + 12 \cdot \frac{1}{y} = 1. \end{cases} \tag{1}$$

To solve this system of equations, it is not necessary to reduce each equation to the form, where the left-hand member is represented by an algebraic fraction, and the right-hand member by zero. In this case it would make the solution more difficult. Here, it is better to regard $\frac{1}{x}$ and $\frac{1}{y}$ as new unknowns. Then we solve System (1) as a linear system of two equations in two unknowns. From the first equation of System (1) we express $\frac{1}{y}$ in terms of $\frac{1}{x}$:

$$\frac{1}{y} = \frac{1}{8} - \frac{11}{8} \cdot \frac{1}{x}, \tag{2}$$

and substitute the expression $\left(\frac{1}{8} - \frac{11}{8} \cdot \frac{1}{x}\right)$ for $\frac{1}{y}$ into the second equation of System (1). We get the equation

$$9 \cdot \frac{1}{x} + 12 \left(\frac{1}{8} - \frac{11}{8} \cdot \frac{1}{x}\right) = 1,$$

or

$$\frac{1}{2} - \frac{15}{2} \cdot \frac{1}{x} = 0,$$

whence $\frac{1}{x} = \frac{1}{15}$. Substituting $\frac{1}{15}$ for $\frac{1}{x}$ into Expression (2), we find $\frac{1}{y} = \frac{1}{30}$. Now it is clear that $x = 15$, and $y = 30$.

Consequently, the first team would complete the job for 15 days, the second for 30 days.

Problem 2. Two points move along a circle in the same direction. The circumference of the circle is equal to 24 metres. The first point completes the circle by 9 minutes faster than the second and overtakes the other point every four minutes. Determine the velocities of these points.

Solution. Let the velocity of the first point be equal to x metres per minute, and the velocity of the second to y metres per minute. Then the first point runs round the circle completing it for $\frac{24}{x}$ minutes, and the second for $\frac{24}{y}$ minutes. Since the first point completes the circle by 9 minutes faster than the second, we have

$$\frac{24}{y} = \frac{24}{x} + 9.$$

The second condition means that for 4 minutes the first point covers the path by 24 metres longer than the second. But during 4 minutes the first point covers $4x$ metres, and the second $4y$ metres therefore

$$4x = 4y + 24.$$

Thus, we have obtained the system of two rational equations in two unknowns:

$$\begin{cases} \frac{24}{y} = \frac{24}{x} + 9, \\ 4x = 4y + 24. \end{cases} \tag{3}$$

From the second equation of this system we express x in terms of y:

$$x = y + 6, \tag{4}$$

and substitute the expression $(y + 6)$ for x into the first equation. We get the equation

$$\frac{24}{y} = \frac{24}{y+6} + 9. \tag{5}$$

Transposing all the terms to the right-hand side and subtracting then the algebraic fractions, we get the equation

$$\frac{-9y^2 - 54y + 144}{y(y+6)} = 0, \tag{6}$$

which is equivalent to Equation (5). We now solve the equation

$$-9y^2 - 54y + 144 = 0,$$

or

$$y^2 + 6y - 16 = 0. \tag{7}$$

Its discriminant

$$D = b^2 - 4ac = 6^2 - 4 \cdot 1 \cdot (-16) = 100 > 0.$$

Consequently, Equation (7) has two roots:

$$y_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-6\pm 10}{2},$$

that is, $y_1 = 2$ and $y_2 = -8$.

Since the numbers y_1 and y_2 do not make the denominator of the left-hand member of Equation (6) vanish, they are its roots. Now, substituting y_1 and y_2 into Expression (4), we find: $x_1 = 8$, $x_2 = -2$.

Thus, System (3) has two solutions:

$$x_1 = 8, \quad y_1 = 2 \quad \text{and} \quad x_2 = -2, \quad y_2 = -8.$$

In our considerations the velocities x and y are positive numbers, therefore the second solution is rejected.

Thus, the velocity of the first point is 8 m.p.m., of the second 2 m.p.m.

Problem 3. It takes each of three workers definite time to complete a certain job. It is known that the third worker fulfils it an hour earlier than the first. Working together, the three workers fulfil the job during an hour. And if the first worker will work during an hour and then cease the work, and then the second worker will work for 4 hours, then both of them together will fulfil the whole scope of work. How much time does it take each worker to carry out the job?

Solution. Suppose the first worker is able to carry out the entire job for x hours, the second for y hours, and the third for z hours. Then during one hour the first worker will do $\frac{1}{x}$ part of the total amount of work, the second $\frac{1}{y}$ part, and the third $\frac{1}{z}$ part. Working together, they fulfil $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ part of the total amount of work per hour; by the hypothesis, it takes them an hour to complete the job, consequently,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

If the first worker will work during an hour, and then the second during 4 hours, they will together do $\left(\frac{1}{x}+4\cdot\frac{1}{y}\right)$ part of the work. But, by the hypothesis, working so, they will fulfil the entire job, consequently,

$$\frac{1}{x}+4\cdot\frac{1}{y}=1.$$

Since the third worker completes the job an hour earlier than the first, we have

$$x = z + 1.$$

Thus, we have obtained the system of three rational equations in three unknowns:

$$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1, \\ \frac{1}{x}+4\cdot\frac{1}{y}=1, \\ x=z+1. \end{cases} \tag{8}$$

From the third equation we express z in terms of x:

$$z = x - 1. \tag{9}$$

From the second equation we express $\frac{1}{y}$ in terms of x:

$$\frac{1}{y}=\frac{1}{4}\cdot\left(1-\frac{1}{x}\right). \tag{10}$$

Substituting Expression (9) and (10) into the first equation of System (8), we get the equation

$$\frac{1}{x}+\frac{1}{4}\cdot\left(1-\frac{1}{x}\right)+\frac{1}{x-1}=1. \tag{11}$$

Transposing all of its terms to the left-hand side and adding the algebraic fractions, we get the equation

$$\frac{3x^2-10x+3}{4x(x-1)}=0, \tag{12}$$

which is equivalent to Equation (11). Let us now solve the equation

$$3x^2 - 10x + 3 = 0. \tag{13}$$

Its discriminant

$$D = b^2 - 4ac = (-10)^2 - 4\cdot3\cdot3 = 64 > 0.$$

Consequently, Equation (13) has two roots:

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{10\pm8}{6},$$

that is, $x_1=3$, $x_2=\frac{1}{3}$.

Since x_1 and x_2 do not make the denominator of the left-hand member of Equation (12) vanish, x_1 and x_2 are the roots of Equation (12). Substituting x_1 and x_2 into (9) and (10), we find:

$$y_1 = 6, \quad y_2 = -2; \quad z_1 = 2, \quad z_2 = -\frac{2}{3}.$$

Consequently, System (8) has two solutions:

$$x_1 = 3, \quad y_1 = 6, \quad z_1 = 2;$$
$$x_2 = \frac{1}{3}, \quad y_2 = -2, \quad z_2 = -\frac{2}{3}.$$

Since x, y and z denote the time (in hours) required to do the work, they cannot be represented by negative numbers. Therefore the problem has the only solution:

$$x_1 = 3, \quad y_1 = 6, \quad z_1 = 2.$$

Thus, the first worker can do the work during 3 hours, the second for 6 hours, and the third for 2 hours.

SECTION TEST

EXERCISES

1. A motorcyclist started from the point A towards the point B the distance between which is equal to 525 km. After a while, a car left B for A and soon met the motorcyclist at the instant when he had covered $\frac{3}{7}$ of the distance from A to B. The motorcyclist and the car continued their drive, and the motorcyclist reached the point B 3 hours after the car's arrival at the point A. If the car left B 1.5 hours earlier than it actually did, then it would meet the motorcyclist at the point 180 km distant from the point A. Find the speeds of the car and motorcyclist assuming them to be constant; determine also how long after the motorcyclist's start the car left B for A.

2. A geologist went uphill. When covering the second half of his way, he reduced his speed by half a kilometre per hour as compared with the speed with which he walked the first half of his way. It took him 12 hours to reach the summit. When he went downhill, using the same way, his speed was by a kilometre per hour higher as compared with the speed he covered the first half of his way uphill. It took him 4 hours to complete his way back, downhill. Find the length of the path he covered and the speed with which he went covering the first half of his way uphill.

3. A river has a tributary. A steamer puts out of the landing-stage A on the tributary, goes with the stream covering 80 km to its mouth to reach the river, then the steamer goes upstream to the landing-stage B. It takes the steamer 18 hours to cover the whole path from A to B. Then it returns back. The time necessary to return from B to A, using the same way, is equal to 15 hours. The speed of the steamer in stagnant water equals 18 km/h, the rate of flow of the river being equal to 3 km/h. Find the distance from A to B and the rate of flow of the tributary.

Chapter 17

QUADRATIC TRINOMIAL

Sec. 17.1. Factoring a Quadratic Trinomial

The function

$$y = ax^2 + bx + c, \tag{1}$$

where a, b, and c are given numbers, and $a \neq 0$, is called a *quadratic trinomial*. This function is defined for all real values of x.

If it is required to find out the values of x for which Function (1) vanishes, then it is necessary to set $y = 0$ in Formula (1) and to solve the quadratic equation

$$ax^2 + bx + c = 0. \tag{2}$$

The roots of this equation, if any, just make the quadratic trinomial (1) vanish.

We know that if the condition

$$b^2 - 4ac \geqslant 0 \tag{3}$$

is fulfilled, then Equation (2) has two real roots x_1 and x_2 which are different for $b^2 - 4ac > 0$ and coincident for $b^2 - 4ac = 0$. And if Condition (3) is not fulfilled, then Equation (2) has no real roots.

The number $D = b^2 - 4ac$ is customarily said to be the discriminant of not only the equation $ax^2 + bx + c = 0$, but also of the quadratic trinomial $ax^2 + bx + c$.

Similarly, the root of the equation $ax^2 + bx + c = 0$ is said to be the *root of the quadratic trinomial* $ax^2 + bx + c$.

Theorem. *The quadratic trinomial*

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

on condition that

$$D = b^2 - 4ac \geqslant 0$$

is factored into factors

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are the roots of this trinomial which are distinct for $D > 0$ and coincident for $D = 0$.

Proof. Factoring out the number a, which is, by hypothesis, different from zero, we get

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right). \tag{4}$$

Separating a perfect square from the trinomial $x^2 + \frac{b}{a}x + \frac{c}{a}$ and using the formula for the difference between two squares, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$$

$$= \left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{D}}{2a}\right)^2 = \left(x + \frac{b}{2a} + \frac{\sqrt{D}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{D}}{2a}\right). \tag{5}$$

Note that the roots of the quadratic trinomial $ax^2 + bx + c$ are computed by the formulas

$$x_1 = -\frac{b+\sqrt{D}}{2a}, \quad x_2 = -\frac{b-\sqrt{D}}{2a}.$$

From (4) and (5) there follows the equality

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

which was required to be proved.

Example 1. Factor the quadratic trinomial

$$x^2 - 5x + 6$$

into two binomial factors.

In this case

$$D = b^2 - 4ac = 25 - 24 = 1 > 0,$$

and therefore the trinomial has two real roots:

$$x_{1,2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6} = \frac{5}{2} \pm \frac{1}{2},$$

that is, $x_1 = 2$, $x_2 = 3$.

Consequently,

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

Example 2. Factor the quadratic trinomial

$$2x^2 - 16x + 30.$$

We have

$$D = b^2 - 4ac = 256 - 240 = 16 > 0.$$

Consequently, the trinomial $2x^2 - 16x + 30$ has two real roots:

$$x_{1,2} = \frac{16 \pm \sqrt{256 - 240}}{4} = \frac{16 \pm 4}{4} = 4 \pm 1,$$

that is,

$$x_1 = 5, \quad x_2 = 3.$$

Therefore

$$2x^2 - 16x + 30 = 2(x - 5)(x - 3).$$

Remark 1. Proved for the condition $b^2 - 4ac \geqslant 0$ the equality

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \tag{6}$$

where x_1 and x_2 are the roots of the trinomial $ax^2 + bx + c$, is true for any values of x, that is, this equality is an identity.

Remark 2. Let a, x_1, and x_2 are given real numbers, and $a \neq 0$. Then the equation

$$a(x - x_1)(x - x_2) = 0 \tag{7}$$

has two roots: $x = x_1$ and $x = x_2$, which are distinct if $x_1 \neq x_2$, and coincident if $x_1 = x_2$.

Indeed, the values $x = x_1$ and $x = x_2$ satisfy, obviously, Equation (7). On the other hand, if the number x is different from x_1 and x_2, then it, obviously, does not satisfy this equation.

Hence, in particular, it follows that under the condition $b^2 - 4ac < 0$, Identity (6), where x_1 and x_2 are real numbers, is impossible since under this condition the equation $ax^2 + bx + c = 0$ has no real roots, whereas the equation $a(x - x_1)(x - x_2) = 0$ would have real roots.

SECTION TEST

QUESTIONS

1. What is a quadratic trinomial?
2. What is the discriminant of a quadratic trinomial?
3. What is the root of a quadratic trinomial?
4. How many roots may a quadratic trinomial have?
5. Under what condition can a quadratic trinomial be factored?

EXERCISES

1. Factor the given quadratic trinomials if it is possible, if not, then explain why:
(a) $x^2 + 8x + 15$; (b) $4x^2 - 4x + 1$; (c) $2x^2 - 3x + 4$.

Sec. 17.2. Vieta's Theorem

The function

$$y = x^2 + px + q, \tag{1}$$

where p and q are given numbers, is called a *reduced quadratic trinomial.*

The quadratic trinomial (1) is a particular case of the general trinomial

$$y = ax^2 + bx + c, \tag{2}$$

where

$$a = 1, \quad b = p, \quad c = q.$$

The discriminant of Trinomial (1)

$$D = b^2 - 4ac = p^2 - 4q.$$

If $D \geqslant 0$, then the quadratic trinomial (1) has two roots computed by the formula

$$x_{1,2} = \frac{-p \pm \sqrt{p^2-4q}}{2},$$

which is usually written in the form

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For $D > 0$ this formula yields two distinct roots of the quadratic trinomial (1), and for $D = 0$ two coinciding roots (or one double root):

$$x_1 = x_2 = -\frac{p}{2}.$$

And if $D < 0$, then, as we know, Equation (1) has no real roots.

Vieta's Theorem. *If a reduced quadratic trinomial has a nonnegative discriminant, then the sum of the roots of this trinomial is equal to its coefficient of x taken with reverse sign, and the product of the roots is equal to its constant term.*

In other words, if x_1 and x_2 are the roots of Trinomial (1), *then*

$$x_1 + x_2 = -p, \quad x_1 x_2 = q. \tag{3}$$

Formulas (3) are called *Vieta's formulas.*

Proof. The roots of Trinomial (1) are computed by the formulas

$$x_1 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}, \quad x_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}. \tag{4}$$

Adding together Equalities (4), we get

$$x_1 + x_2 = \left(-\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}\right) + \left(-\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\right) = -p.$$

Multiplying Equalities (4) by each other and applying the formula for the difference between two squares, we get

$$\begin{aligned} x_1 x_2 &= \left(-\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}\right)\left(-\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\right) \\ &= \left(-\frac{p}{2}\right)^2 - \left(\sqrt{\left(\frac{p}{2}\right)^2 - q}\right)^2 = \frac{p^2}{4} - \frac{p^2}{4} + q = q. \end{aligned}$$

Vieta's theorem has been proved.

Note that for the general quadratic trinomial

$$y = ax^2 + bx + c \qquad (a \neq 0) \tag{5}$$

Vieta's theorem is formulated as follows: *if the discriminant of the quadratic trinomial* (5) *is nonnegative, and if* x_1 *and* x_2 *are the roots of Trinomial* (5), *then*

$$x_1+x_2=-\frac{b}{a}, \qquad x_1x_2=\frac{c}{a}.$$

Its proof is carried out exactly in the same way as the proof of Vieta's theorem for a reduced quadratic trinomial.

Vieta's theorem is often used in solving various problems.

Example 1. Write the reduced quadratic trinomial whose roots are the numbers 1 and -3. In other words, find numbers p and q such that the quadratic trinomial

$$y = x^2 + px + q$$

has the roots $x_1 = 1$ and $x_2 = -3$.

By Vieta's formulas,

$$-p = x_1 + x_2 = -2, \quad q = x_1x_2 = -3.$$

Hence, the desired quadratic trinomial has the form

$$y = x^2 + 2x - 3.$$

Example 2. Determine by inspection the signs of the roots of the quadratic trinomial

$$y = x^2 - 364x + 497. \tag{6}$$

Since the discriminant of this trinomial

$$D = (364)^2 - 4\cdot 497$$

is positive, the trinomial has two distinct roots x_1 and x_2. By Vieta's theorem,

$$x_1x_2 = 497,$$

that is, the product of the roots is positive. Hence, the roots have like signs, that is, either $x_1 > 0$ and $x_2 > 0$, or $x_1 < 0$ and $x_2 < 0$.

Also, by Vieta's theorem,

$$x_1 + x_2 = 364,$$

that is, the sum of the roots is also positive.

Hence, the quadratic trinomial (6) has two positive roots.

SECTION TEST

QUESTIONS

1. What is a reduced quadratic trinomial?
2. What is the formulation of Vieta's theorem?
3. Which formulas are called Vieta's formulas?

EXERCISES

1. Write the reduced quadratic trinomial whose roots are the indicated numbers:

(a) $x_1=0$, $x_2=-1$, (b) $x_1=4$, $x_2=-\frac{3}{4}$, (c) $x_1=-1+\sqrt{7}$, $x_2=-1-\sqrt{7}$.

2. Determine, by inspection, the signs of the roots of the quadratic trinomial: (a) $x^2+1000x+2139$, (b) $x^2-200x-39$.

Sec. 17.3. The Function $y=ax^2$ $(a>0)$

The function

$$y = ax^2, \tag{1}$$

where a is a given positive number, is defined for any real values of x. Its properties are very much in common with those of the function

$$y = x^2, \tag{2}$$

considered in Sec. 10.7. Let us mention them:

(1) if $x = 0$, then $y = 0$;

(2) if $x \neq 0$, then $y > 0$;

(3) for nonnegative values of x Function (1) increases, for nonpositive values of x it decreases;

(4) $y \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$;

(5) Function (1) is even;

(6) Function (1) is continuous.

These properties can be proved in the same way as they were proved in Chapter 10 (Sec. 10.7) for Function (2).

The graph of Function (1) resembles very much the graph of the Parabola (2).

The curve representing the graph of Function (1) is also called a *parabola*. We shall say briefly: "parabola $y = ax^2$".

Example 1. Consider two functions: $y = x^2$ and $y = 2x^2$. Both of them are defined for any real values of x.

Let us introduce the Cartesian coordinate system Oxy, and let a certain point A (x_0, y_0) belong to the graph of the parabola $y = x^2$, that is, let $y_0 = x_0^2$ (Fig. 27).

Let us take the point A_1 $(x_0, 2y_0)$. Its coordinates satisfy the condition

$$2y_0 = 2x_0^2,$$

therefore the point A_1 $(x_0, 2y_0)$ belongs to the graph of the parabola $y = 2x^2$. The ordinates of the points A_1 and A are in the ratio 2 : 1, that is, the point A_1 is obtained from the point A by stretching the

ordinate of the point A twice, and the point A is obtained from the point A_1 by contracting the ordinate of the point A_1 by half. This reasoning is applicable to any points of the graphs of the functions $y = x^2$ and $y = 2x^2$ having the same abscissa x. Therefore the graph of the parabola $y = 2x^2$ is said to be obtained from the graph of the function $y = x^2$ by stretching the latter twice along the y-axis.

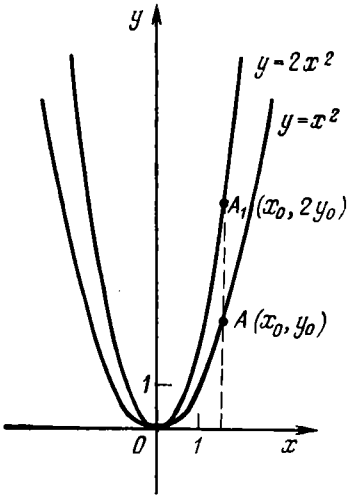


Fig. 27

We see (Fig. 27) that the graphs of the parabolas $y = 2x^2$ and $y = x^2$ are in fact very much alike.

Reasoning in a similar way, we can show that the graph of the parabola

$$y = ax^2,$$

where a is a given number exceeding unity, is obtained from the graph of the parabola $y = x^2$ by stretching the latter a-fold along the y-axis.

Figure 28 represents the parabolas $y = x^2$, $y = 2x^2$, and $y = 3x^2$ in one and the same Cartesian coordinate system Oxy.

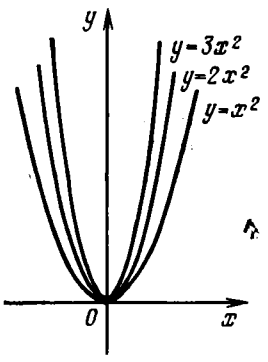


Fig. 28

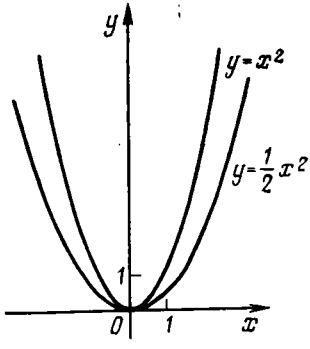


Fig. 29

Example 2. Consider now two functions:

$$y = x^2 \quad \text{and} \quad y = \frac{1}{2}x^2.$$

Reasoning as above, we get that the graph of the parabola $y = \frac{1}{2}x^2$ is obtained from the graph of the parabola $y = x^2$ contracting the latter by half along the y-axis (Fig. 29).

Exactly in the same way, the graph of the parabola

$$y = ax^2,$$

where a is a positive number less than unity, is obtained from the graph of the parabola $y = x^2$ by contracting the latter $\frac{1}{a}$ times along the y-axis.

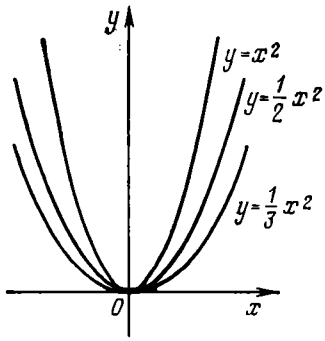


Fig. 30

Figure 30 represents the parabolas $y = x^2$, $y = \frac{1}{2}x^2$, and $y = \frac{1}{3}x^2$.

SECTION TEST

QUESTIONS

1. What is the name of the curve which is the graph of the function $y = 5x^2$?
2. What line is the axis of symmetry of the parabola $y = ax^2$ $(a > 0)$?

EXERCISES

1. Draw in the Cartesian coordinate system Oxy the graphs of the parabolas: (a) $y = 0.3x^2$, (b) $y = 3x^2$, (c) $y = 1.5x^2$.
2. Formulate and prove the properties of the function $y = 2x^2$.

Sec. 17.4. The Function $y = ax^2$ $(a \neq 0)$

Figure 31 represents the parabolas

$$y = x^2; \quad y = -x^2.$$

The first of them is situated above and the second below the x-axis if, of course, the point O (0, 0) is not taken into consideration. These parabolas are symmetric with respect to the x-axis, that is, the

ordinates of their points, having the same abscissa x, have unlike signs.

Similarly, the parabolas

$$y = ax^2; \quad y = -ax^2,$$

where a is a given nonzero number, are symmetric with respect to the x-axis. For $a > 0$ the first of them is situated above, and the

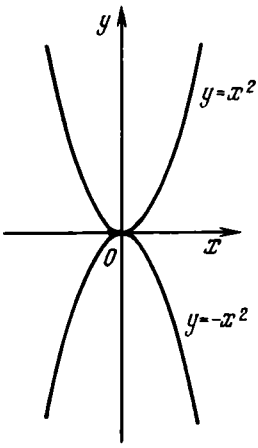


Fig. 31

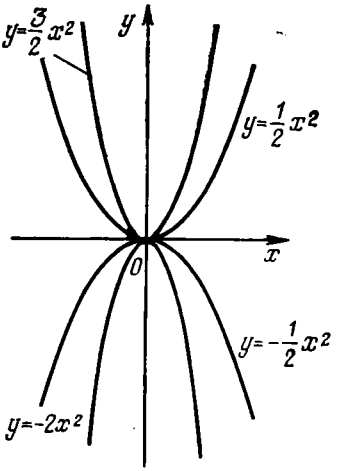


Fig. 32

second below the x-axis if the point O (0, 0) is excluded from consideration.

Figure 32 represents in the same Cartesian system Oxy the parabolas

$$y = \frac{3}{2}x^2; \quad y = \frac{1}{2}x^2; \quad y = -\frac{1}{2}x^2; \quad y = -2x^2.$$

We should also like to note that for any given value of a ($a \neq 0$) the parabola

$$y = ax^2$$

is symmetric with respect to the y-axis since for any x the equality

$$ax^2 = a(-x^2).$$

Thus, the y-axis serves as the axis of symmetry of the parabola $y = ax^2$ ($a \neq 0$).

The point at which the parabola $y = ax^2$ intersects its axis of symmetry is called the *vertex* of the parabola.

The axis of symmetry divides the parabola into two parts which are called the *branches* of the parabola. If a is a positive number, then the parabola $y = ax^2$, opens upward and if a is negative, then the parabola opens downward.

SECTION TEST

QUESTIONS

1. What is the name of the curve representing the graph of the function $y = ax^2$ ($a \neq 0$)?
2. What point serves as the vertex of the parabola $y = ax^2$?
3. What line is the axis of symmetry of the parabola $y = ax^2$?

EXERCISES

1. Draw in the Cartesian coordinate system Oxy the graphs of the parabolas: (a) $y = -3x^2$, (b) $y = -\frac{1}{4}x^2$.

2. Draw in one and the same Cartesian coordinate system Oxy the graphs of the following parabolas: $y = 3x^2$, $y = -3x^2$, $y = \frac{1}{3}x^2$, $y = -\frac{1}{3}x^2$, $y = x^2$, $y = -x^2$.

Sec. 17.5. The Function $y = a(x - x_0)^2 + y_0$

In the coordinate plane Oxy, let us shift the parabola $y = ax^2$ parallel to itself, i.e. so that its axis of symmetry remains parallel to itself all the time. After such a shift its equation will change.

Figure 33 represents the parabola $y = ax^2$ for $a > 0$ and the parabola obtained as a result of translation of the first parabola.

The vertex of the second parabola, the point D, has the coordinates (x_0, y_0). Marked on the second parabola is an arbitrary point $B(x, y)$.

Let B_1 be the projection of the point B on the x-axis, and let the straight line DC be parallel to the x-axis.

In Figure 33

$$x = OB_1, \quad y = BB_1,$$
$$DC = x - x_0, \quad BC = y - y_0.$$

Besides, it is obvious that $BC = a \cdot (DC)^2$, therefore

$$y - y_0 = a(x - x_0)^2,$$

or

$$y = a(x - x_0)^2 + y_0. \qquad (1)$$

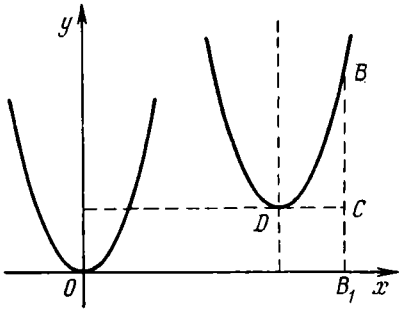


Fig. 33

We have obtained that the arbitrary point B of the second parabola has the coordinates (x, y) satisfying Equation (1).

Thus, it has been proved that if the parabola $y = ax^2$ is translated in the plane Oxy so that its vertex turns out to be at the point (x_0, y_0), then the translated parabola has Equation (1).

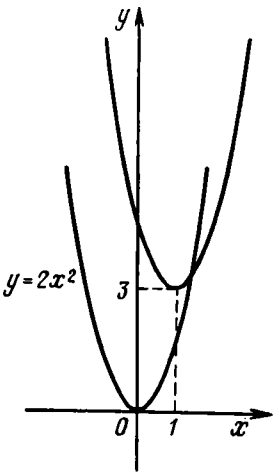


Fig. 34

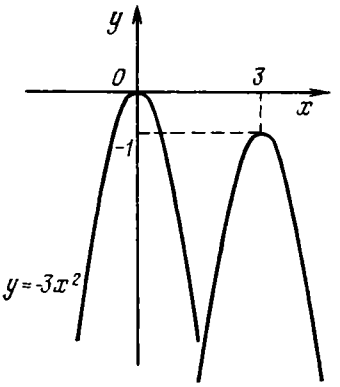


Fig. 35

Remark. Formula (1) was derived using Fig. 33, where it was supposed that $a > 0$, $x_0 > 0$, and $y_0 > 0$. But this formula also remains true for any x_0, y_0, and a ($a \neq 0$).

Example 1. The equation

$$y = 2(x - 1)^2 + 3$$

is the equation of the parabola obtained from the parabola $y = 2x^2$ if the latter is translated so that its vertex turns out to be found at the point (1, 3) (Fig. 34).

Example 2. If the parabola

$$y = -3x^2$$

is translated so that the new coordinates of its vertex are (3, -1), then its equation will be (Fig. 35)

$$y \overset{\curvearrowleft}{=} -3(x-3)^2 - 1.$$

SECTION TEST

QUESTIONS

1. How do we obtain the graph of the parabola $y = a(x - x_0)^2 + y_0$ from the graph of the parabola $y = ax^2$?

EXERCISES

1. Write the equation of the parabola obtained from the parabola (a) $y = x^2$; (b) $y = -x^2$; (c) $y = 2x^2$; (d) $y = -2x^2$ by translating its vertex to the point (1, -1).
2. In the plane Oxy draw the following parabolas:
(a) $y = 3(x - 1)^2 + 1$; (b) $y = -(x + 1)^2 - 1$;
(c) $y = 2x^2 + 2$; (d) $y = 2(x - 1)^2 + 1$.

Sec. 17.6. The Graph of a Quadratic Trinomial

The function

$$y = ax^2 + bx + c, \tag{1}$$

where a, b, and c are given numbers, and $a \neq 0$, is called, as we know, a quadratic trinomial.

The function $y = ax^2$ is a particular case of Function (1) for $b = c = 0$. Its graph in the Cartesian coordinate system Oxy is the parabola

$$y = ax^2. \tag{2}$$

The examples given below will show that the graph of Trinomial (1) is simply the parabola $y = ax^2$ translated in the plane Oxy to the appropriate place.

Example 1. Construct the graph of the quadratic trinomial

$$y = x^2 - 2x - 3. \tag{3}$$

Let us isolate a perfect square:

$$x^2 - 2x - 3 = x^2 - 2x + 1 - 4 = (x - 1)^2 - 4.$$

Thus, Trinomial (3) can be written in the following form:

$$y = (x - 1)^2 - 4. \tag{4}$$

But, then, by virtue of the aforegoing, the graph of Trinomial (3) is the parabola

$$y = x^2$$

translated so that its vertex is the point (1, —4) (Fig. 36).

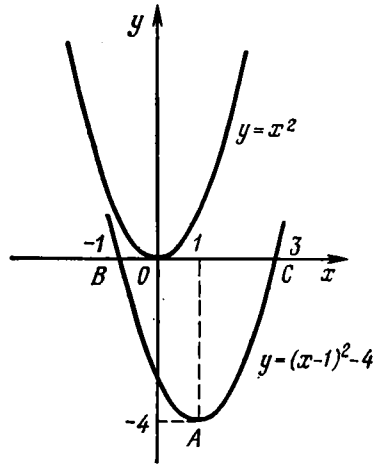


Fig. 36

Since the coefficient of x^2 in Equation (3) is positive, Parabola (3) opens upward. Its vertex is found below the x-axis. This indicates obviously that the parabola intersects the x-axis at two points, that is, if we set $y = 0$ in Equality (3), then the obtained quadratic equation

$$x^2 - 2x - 3 = 0$$

must have two real roots.

Of course, this conclusion can be verified by computing the roots of this equation.

We have

$$x_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2,$$

that is, $x_1 = -1$, $x_2 = 3$.

If we wish to draw the graph of the quadratic trinomial (3) with a sufficient accuracy, then it is best of all to draw the parabola $y = x^2$ in the coordinate system Oxy and then to translate it so that the translated parabola has its vertex at the point (1, —4).

A schematic graph of Parabola (3) can be obtained on the basis of the following facts.

We mark in the plane Oxy the point A (1, —4), the vertex of Parabola (3), its axis, a straight line passing through the point A, and the points B (—1, 0) and C (3, 0) at which Parabola (3) intersects the x-axis. The schematic graph of Parabola (3) represents a smooth curve passing through the points A, B, and C and symmetric with respect to the marked axis. This curve opens upward.

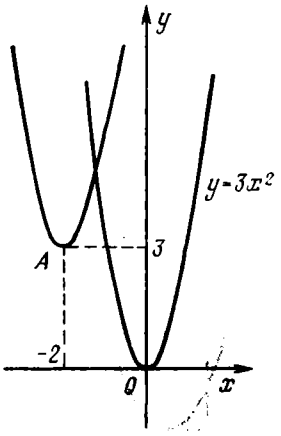


Fig. 37

Example 2. Construct the graph of the quadratic trinomial

$$y = 3x^2 + 12x + 15. \qquad (5)$$

Factoring out the coefficient 3 and isolating a perfect square from the obtained trinomial, we get

$$3(x^2 + 4x + 5) = 3(x^2 + 2 \cdot 2x + 4 + 1) = 3(x + 2)^2 + 3.$$

Thus, Trinomial (5) may be written in the following form:

$$y = 3(x + 2)^2 + 3.$$

Therefore the graph of Trinomial (5) is the parabola

$$y = 3x^2,$$

translated so that the vertex of the translated parabola is the point A (—2, 3) (Fig. 37).

As is seen from the graph, the ordinate of Function (5) is positive for any value of x, that is,

$$3x^2 + 12x + 15 > 0$$

for all x's. This means that the equation

$$3x^2 + 12x + 15 = 0$$

has no real roots, and, consequently, the discriminant of the quadratic trinomial (5) is negative.

Indeed,

$$D = b^2 - 4ac = 144 - 180 = -36 < 0.$$

To draw a schematic graph of Parabola (5), we have to plot in the plane Oxy at least several of its points. The vertex A (—2, 3)

is a very important point for constructing the graph of the given quadratic trinomial. We can also assign, for instance, the values $x = -1$ and $x = 0$ and compute the corresponding values of y using Formula (5):

$$x = -1, \quad y = 3 - 12 + 15 = 6;$$
$$x = 0, \quad y = 15.$$

Through the three points thus obtained we can approximately draw the right-hand branch of Parabola (5) and then complete the graph by drawing its left-hand branch symmetric to the right-hand branch with respect to the vertical line passing through the point A.

Example 3. Construct the graph of the quadratic trinomial

$$y = -6x^2 - 12x - 8. \qquad (6)$$

Let us transform the given trinomial:

$$-6x^2 - 12x - 8 = -6(x^2 + 2x) - 8$$
$$= -6(x^2 + 2x + 1) + 6 - 8 = -6(x + 1)^2 - 2.$$

Consequently, it can be written in the form

$$y = -6(x + 1)^2 - 2,$$

and, therefore, the graph of Function (6) is the parabola

$$y = -6x^2$$

translated so that the vertex of the translated parabola turns out to be at the point A $(-1, -2)$ (Fig. 38).

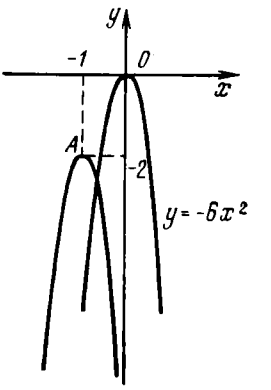


Fig. 38

We see that this time the parabola does not intersect the x-axis either, and the equation

$$-6x^2 - 12x - 8 = 0$$

has no real roots; the inequality

$$-6x^2 - 12x - 8 < 0$$

is fulfilled for all real x's.

SECTION TEST

QUESTIONS

1. How do we construct the graph of the quadratic trinomial $y = ax^2 + bx + c$?

EXERCISES

1. Plot the graph of the quadratic trinomial:
(a) $y = 2x^2 - 4x + 3$; (b) $y = -3x^2 - 3x + 18$;
(c) $y = x^2 - 10x + 25$; (d) $y = -x^2 + 2x - 3$.

Chapter 18

THE DERIVATIVE OF A POLYNOMIAL

Sec. 18.1. Instantaneous Velocity

Example. Let a point move along the s-axis according to the rule

$$s = 5t^2. \tag{1}$$

At the instant of time t the coordinate s of the point is determined by Equality (1). At the instant $t + \Delta t$ the corresponding value of the coordinate $s + \Delta s$ is determined by the equality

$$s + \Delta s = 5\,(t + \Delta t)^2.$$

The path covered by the point during the time interval $[t, t + \Delta t]$ is equal to

$$\Delta s = 5\,(t + \Delta t)^2 - 5t^2,$$

that is, to the increment of the function $s = 5t^2$ at the point t which corresponds to the increment of the argument Δt.

The average velocity of the point during the time interval $[t, t + \Delta t]$ is equal to

$$\frac{\Delta s}{\Delta t} = \frac{5\,(t+\Delta t)^2 - 5t^2}{\Delta t},$$

that is, to the ratio of the increment Δs of the function $s = 5t^2$ at the point t to the corresponding increment Δt of the argument.

Since

$$\Delta s = 5\,(t^2 + 2t\,\Delta t + \Delta t^2 - t^2) = 10t\,\Delta t + 5\Delta t^2,$$

the indicated average velocity is equal to

$$\frac{\Delta s}{\Delta t} = 10t + 5\,\Delta t. \tag{2}$$

We see that the average velocity depends on t and Δt. This means that the motion defined by the rule $s = 5t^2$ is nonuniform, that is, the average velocity for different time intervals $[t, t + \Delta t]$ is, generally speaking, different.

It is seen from Formula (2) that for a given value of t for small Δt the average velocity is approximately equal to the number $10t$:

$$\frac{\Delta s}{\Delta t} = 10t + 5\,\Delta t \cong 10t.$$

More exactly, if the increment Δt tends to zero, then the average velocity $\frac{\Delta s}{\Delta t}$ tends to the number $10t$. We say that the limit $\frac{\Delta s}{\Delta t}$, as $\Delta t \to 0$, is equal to $10t$ and write:

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = 10t.$$

The limit to which the average velocity of motion tends during the time interval $[t, t + \Delta t]$ as $\Delta t \to 0$ is called the *instantaneous velocity of motion at the time t* or, simply, the *velocity of motion at the time t* and is usually denoted by the letter v.

In the general case we may consider the motion of a point along the s-axis expressed by the rule

$$s = f(t),$$

where f is a given function of t.

The distance travelled by this point during the time interval $[t, t + \Delta t]$ is equal to the increment

$$\Delta s = f(t + \Delta t) - f(t)$$

of the function f at the point t corresponding to the increment of the argument Δt.

The average velocity of the point during the time interval $[t, t + \Delta t]$ is equal to the ratio of the increment of the function f at the point t to the corresponding increment of the argument Δt:

$$v_{av} = \frac{\Delta s}{\Delta t}.$$

Finally, *the instantaneous velocity v at the instant t of a point moving according to the rule $s = f(t)$ is equal to the limit of the ratio of the increment of the function f at the point t to the corresponding increment of the argument Δt as the latter tends to zero:*

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

SECTION TEST

EXERCISES

1. A point travels along the s-axis according to the rule $s = f(t)$, where the function $f(t)$ is defined by the formula:
(a) $2t^2$; (b) t^3; (c) $6t$; (d) $3t^2 + 2$.
Determine the distance travelled by the point during the time interval $[t, t + \Delta t]$; the average velocity during this time interval; the instantaneous velocity at the instant t and, in particular, at the instant $t = 1$.

Sec. 18.2. Derivative

Let there be given a function

$$y = f(x) \quad (a < x < b)$$

defined on the interval (a, b).

Let us also be given an arbitrary point x of the interval (a, b) and another point $x + \Delta x$ from the same interval. We shall compute the increment of the function $y = f(x)$ at the point x corresponding to the increment of the argument Δx:

$$\Delta y = f(x + \Delta x) - f(x),$$

and take the ratio of Δy to Δx.

The limit of the ratio of the increment Δy of the function f at the point x to the corresponding increment of the argument Δx is called the derivative of the function f at the point x and is denoted as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Or in other words: *The derivative of a function, $y = f(x)$, is defined as the limit of the ratio, $\Delta y / \Delta x$, as the increment Δx approaches zero as a limit.*

On the basis of the preceding section we may say that the *instantaneous velocity at the instant t of the point moving according to the rule $s = f(t)$ is equal to the derivative of the function f at the point t:*

$$v = f'(t).$$

Given below are several important examples of computing derivatives but prior to this we are going to formulate a number of properties of limits which we shall regard as obvious and, therefore, will not prove them:

the limit of the sum of two variables is equal to the sum of their limits

$$\lim (\alpha + \beta) = \lim \alpha + \lim \beta;$$

the limit of the difference between two variables is equal to the difference between their limits:

$$\lim (\alpha - \beta) = \lim \alpha - \lim \beta;$$

the limit of the product of two variables is equal to the product of their limits:

$$\lim (\alpha\beta) = \lim \alpha \cdot \lim \beta;$$

the limit of the constant C is equal to this constant:

$$\lim C = C.$$

(1) The derivative of the constant $y = C$ is equal to zero:

$$(C)' = 0. \tag{1}$$

The constant $y = C$ may be considered as a function of x equal for all values of x to one and the same value of C.

Thus, to the value of x there corresponds $y = C$, and to the value of $x + \Delta x$ there corresponds $y + \Delta y = C$, therefore

$$\Delta y = C - C = 0,$$

consequently,

$$\frac{\Delta y}{\Delta x} = 0,$$

and the derivative

$$C' = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right) = \lim_{\Delta x \to 0} 0 = 0.$$

(2) The derivative of the function $y = x$ is equal to 1:

$$x' = 1.$$

Indeed, in this case to the value of x there corresponds $y = x$, and to the value $x + \Delta x$ there corresponds $y + \Delta y = x + \Delta x$. Therefore $\Delta y = \Delta x$, and

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x} = 1.$$

Consequently, the derivative

$$x' = \lim_{\Delta x \to 0} 1 = 1.$$

(3) The derivative of the function $y = x^2$ is equal to $2x$:

$$(x^2)' = 2x.$$

Indeed,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\,\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = \lim_{\Delta x \to 0} 2x + \lim_{\Delta x \to 0} \Delta x = 2x.$$

(4) The derivative of the function $y = x^3$ is equal to $3x^2$.

Indeed,

$$(x^3)' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\,\Delta x + \Delta x^2) = \lim_{\Delta x \to 0} 3x^2 + \lim_{\Delta x \to 0} (3x\,\Delta x) + \lim_{\Delta x \to 0} \Delta x^2$$

$$= 3x^2 + 3x \lim_{\Delta x \to 0} \Delta x + (\lim_{\Delta x \to 0} \Delta x)^2 = 3x^2.$$

(5) The derivative of the function $y = x^n$ is equal to nx^{n-1}:

$$(x^n)' = nx^{n-1}.$$

We give this formula without proof. As an exercise, we recommend the reader to prove it for $n = 4$ and $n = 5$.

Let us dwell on three other important properties of derivatives.

I. *The derivative of the sum of two functions is equal to the sum of the derivatives of these functions*:

$$|f(x) + \varphi(x)|' = f'(x) + \varphi'(x).$$

Indeed,

$$|f(x)+\varphi(x)|' = \lim_{\Delta x \to 0} \frac{|f(x+\Delta x)+\varphi(x+\Delta x)| - |f(x)+\varphi(x)|}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)-f(x)}{\Delta x} + \frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} = f'(x)+\varphi'(x).$$

II. *The derivative of the difference between two functions is equal to the difference between the derivatives of these functions*:

$$|f(x) - \varphi(x)|' = f'(x) - \varphi'(x).$$

The proof is left to the reader.

III. *When computing a derivative, the constant may be factored out*:

$$|Cf(x)|' = Cf'(x).$$

Indeed,

$$|Cf(x)|' = \lim_{\Delta x \to 0} \frac{Cf(x+\Delta x)-Cf(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[C \frac{f(x+\Delta x)-f(x)}{\Delta x} \right]$$

$$= C \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = Cf'(x).$$

On the basis of the aforegoing, there take place, for instance, the following equalities:

$$(ax^2 + bx + c)' = (ax^2)' + (bx)' + c' = 2ax + b,$$
$$(ax^3 + bx^2 + cx + d)' = 3ax^2 + 2bx + c,$$

where a, b, c, d are constant numbers.

SECTION TEST

EXERCISES

1. Find the derivative of the indicated function:
(a) $3x - 2$; (b) $3x^2 + 2x - 1$; (c) $7x^3 - 2x^2 + 3x + 1$;
(d) $x^4 - 1$; (e) $x^5 + x^4 - x^3$.
2. The rule of motion of a point is determined by the functions:
(a) $3t$; (b) $3t^2 + 1$; (c) $5t^3 - 2t^2 + 2$.
Find the velocity of its motion at the instant t, in particular, for $t = 2$.

Sec. 18.3. Primitive Functions or Antiderivatives

The derivative of the function x^3 is equal to $3x^2$:

$$(x^3)' = 3x^2.$$

We also say that x^3 is a *primitive function* for the function $3x^2$. Similarly, we say that $2x^2$ is a primitive function for the function $4x$ since the derivative of $2x^2$ is equal to $4x$.

In general, *the function $F(x)$ is called the primitive function (or primitive) for the function $f(x)$ on the interval (a, b) if the derivative of $F(x)$ is equal to $f(x)$ on this interval*:

$$F'(x) = f(x).$$

Note that if $F(x)$ is an antiderivative for the function $f(x)$, then the sum $F(x) + C$, where C is any constant, is also an antiderivative for $f(x)$ since

$$(F(x) + C)' = F'(x) = f(x).$$

For instance, not only x^3 but also $x^3 + C$ is an antiderivative for $3x^2$, where C is an arbitrary constant.

Let us write the equalities

$$(C)' = 0, \quad x' = 1, \quad \left(\frac{x}{2}\right)' = x, \quad \left(\frac{x^3}{3}\right)' = x^2,$$

$$\left(\frac{x^4}{4}\right)' = x^3, \quad \ldots \quad \left(\frac{x^n}{n}\right)' = x^{n-1}.$$

It follows from these equalities that the functions

$$C; \quad x + C; \quad \frac{x^2}{2} + C; \quad \frac{x^3}{3} + C; \quad \frac{x^4}{4} + C; \quad \ldots; \quad \frac{x^n}{n} + C,$$

where C is an arbitrary constant, are primitives for the respective functions

$$0; \quad 1; \quad x; \quad x^2; \quad x^3; \ldots; \quad x^{n-1} \ldots .$$

But now we can compute primitives for any polynomials in x. For instance, for the polynomial

$$3x^2 + 2x - 2$$

the primitive functions are written in the following way:

$$\frac{3x^3}{3} + 2 \cdot \frac{x^2}{2} - 2x + C = x^3 + x^2 - 2x + C,$$

where C is an arbitrary constant, since

$$(x^3 + x^2 - 2x + C)' = 3x^2 + 2x - 2.$$

And the primitive functions for the polynomial

$$7x^3 - 3x^2 + x - 2$$

are written as follows:

$$\frac{7x^4}{4}-3\cdot\frac{x^3}{3}+\frac{x^2}{2}-2x+C=\frac{7x^4}{4}-x^3+\frac{x^2}{2}-2x+C,$$

where C is an arbitrary constant, since

$$\left(\frac{7x^4}{4}-x^3+\frac{x^2}{2}-2x+C\right)'=7x^3-3x^2+x-2.$$

SECTION TEST

EXERCISES

1. Find the primitive functions for the following polynomials:
(a) 2; (b) $3x-2$; (c) $x+1$; (d) $2x^2+4$; (e) x^2-x+2.

Sec. 18.4. Examples from Physics

Example 1. Galileo released loads from the height of the Pisa Cathedral and measured the distances traversed by them at different instants of time. As a result, he arrived at the following formula:

$$v=gt, \tag{1}$$

where $g=9.8$, t is the time elapsed since the release of a load, and v the velocity of the load in metres per second at the instant t.

Given Formula (1), it is possible to restore the law of motion of the load $s=f(t)$, that is, to find the function expressing this law. Let us try to do this.

Let us direct the s-axis vertically downward and take the point of load release for the origin O. Let

$$s=f(t)$$

be the law of load motion. The derivative of the function $f(t)$ is equal to the velocity of the load at the instant t. Consequently,

$$f'(t)=gt.$$

We have obtained that $f(t)$ is a primitive function for gt. But then

$$f(t)=g\frac{t^2}{2}+C,$$

where C is an unknown (for the time being) constant.

To find C, note that $s=0$ at the initial instant of time $t=0$, therefore

$$C=0.$$

Consequently, the law of load motion is expressed by the function

$$s=\frac{gt^2}{2}.$$

Example 2. Let a freely falling body move with a velocity

$$v = gt + v_0,$$

where t is the time elapsed from the beginning of the fall. It is necessary to find the law of motion

$$s = f(t).$$

Since the derivative $f'(t)$ is equal to the velocity of motion, we have

$$f'(t) = gt + v_0,$$

and $f(t)$ is a primitive function for $gt + v_0$. But then

$$s = f(t) = \frac{gt^2}{2} + v_0 t + C,$$

where C is an unknown constant. Since $s = 0$ at the instant $t = 0$, we get $C = 0$. Therefore, the required rule of motion is determined by the equality

$$s = \frac{gt^2}{2} + v_0 t. \tag{2}$$

Note that for $t = 0$ $v = v_0$. This indicates that at the instant the body begins to fall it has the initial velocity $v = v_0$.

SECTION TEST

EXERCISES

1. For simplicity, we shall assume the acceleration of gravity $g = 10$ m/s^2. At the instant of time $t = 0$ a body located at a height of 15 m above the Earth's surface received an upward velocity of 10 m/s.

Find: (a) the rule of motion; (b) the maximum height reached by the body; (c) the velocity of the body at this instant; (d) the time necessary for the upward motion; (e) the time during which the body moved downward; (f) the velocity with which the body will fall on the ground.

Chapter 19

LINEAR INEQUALITIES IN ONE UNKNOWN

Sec. 19.1. First-degree Inequalities

The general inequality of the first degree in an unknown x has the form

$$kx + b > 0 \tag{1}$$

or

$$kx + b < 0, \tag{2}$$

where k and b are given numbers ($k \neq 0$). The number k is called the coefficient of the unknown, and the number b the constant term of Inequality (1) or (2).

The following inequalities may serve as examples of first-degree inequalities in one unknown x:

$$3x + 2 > 0; \quad -x - 5 > 0; \quad 2x - 7 < 0;$$
$$-5x + 1 < 0; \quad -x > 0.$$

A *solution* of an inequality in one unknown is defined as a number x_0 such that, when substituted for x into the inequality, turns it into a true numerical inequality.

To *solve* an inequality means to find all of its solutions.

Example 1. Solve the inequality

$$2x + 5 < 0 \tag{3}$$

To solve it, we may reason as follows.

Let a certain number x_0 be a solution of Inequality (3). Substituting it for x into Inequality (3), we get a true numerical inequality:

$$2x_0 + 5 < 0.$$

Adding the number (-5) to both sides of this inequality, we get a true numerical inequality:

$$2x_0 < -5$$

Dividing it by the positive number 2, we get the true numerical inequality

$$x_0 < -\frac{5}{2}.$$

This reasoning is applicable to any solution of Inequality (3). Consequently, any solution x of Inequality (3) is a number satisfying the inequality $x < -\frac{5}{2}$.

Let now a number x_1 be such that

$$x_1 < -\frac{5}{2}.$$

Multiplying this numerical inequality by the positive number 2, we get the true numerical inequality

$$2x_1 < -5.$$

Adding the number 5 to both sides, we get the true numerical inequality

$$2x_1 + 5 < 0.$$

Consequently, the number x_1 is a solution of Inequality (3). This reasoning fits for any number x satisfying the inequality $x < -\frac{5}{2}$. Hence, any number x satisfying the inequality $x < -\frac{5}{2}$ is a solution of Inequality (3).

Thus, the set of all solutions of Inequality (3) is the set of numbers satisfying the inequality $x < -\frac{5}{2}$. We may also say that all the solutions of Inequality (3) form the interval $\left(-\infty, -\frac{5}{2}\right)$, or that the set of all solutions of Inequality (3) is the interval $\left(-\infty, -\frac{5}{2}\right)$.

We may proceed in a similar way when solving general first-degree inequalities (1) and (2).

These considerations imply the following method for solving first-degree inequalities in one unknown:

first transpose the constant term of this inequality to the right side (naturally, with the sign of the number b reversed);

then divide both sides of the obtained inequality by the coefficient of the unknown (if $k > 0$, then the sense of the inequality remains unchanged, and if $k < 0$, then its sense is reversed). As a result, we get an inequality which determines all the solutions of the original inequality in an explicit form.

Using this technique, let us solve the inequality

$$-4x + 13 < 0. \tag{4}$$

Transposing the constant term to the right-hand side, we get the inequality

$$-4x < -13.$$

Dividing both sides of this inequality by the negative number (-4), we get the inequality $x > \frac{13}{4}$ (pay attention to the sense of the inequality).

The last inequality gives all the solutions of Inequality (4) in an explicit form: this is the set of numbers satisfying the inequality $x > \frac{13}{4}$. We may also say that the set of all the solutions of Inequality (4) is the interval $\left(\frac{13}{4}, +\infty\right)$.

SECTION TEST

QUESTIONS

1. What is called the first-degree inequality in one unknown?
2. What is a solution of an inequality in one unknown?

3. What does it mean to solve an inequality in one unknown?
4. What method may be used for solving a first-degree inequality in one unknown?

EXERCISES

1. Solve the following inequalities:
(a) $-x + 5 > 0$; (b) $3x + 4 < 0$; (c) $-2x - 7 > 0$.

Sec. 19.2. Applying Graphs to Solving First-degree Inequalities

Let us show how it is possible, by applying the graphical method, to solve inequalities of the form

$$kx + b > 0, \tag{1}$$

or

$$kx + b < 0. \tag{2}$$

where k and b are given numbers, and $k \neq 0$.

Consider the straight line

$$y = kx + b \tag{3}$$

in the Cartesian coordinate system Oxy. Figure 39 shows such a straight line for $k > 0$, and Fig. 40 for $k < 0$.

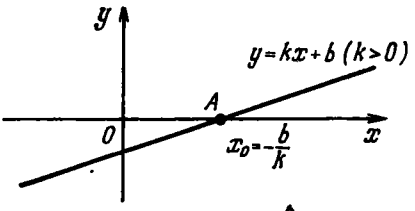


Fig. 39

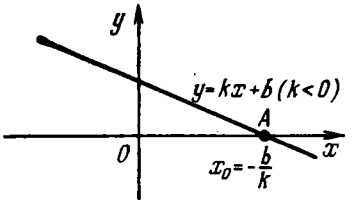


Fig. 40

To solve Inequality (1) means to find all the solutions x for which the line $y = kx + b$ is situated above the x-axis. Here, an important role is played by the point A at which the line (3) intersects the x-axis.

Let x_0 denote the abscissa of the point A. Since its ordinate is equal to zero, x_0 satisfies the equation

$$0 = kx_0 + b,$$

whence

$$x_0 = -\frac{b}{k}.$$

Look at Fig. 39 corresponding to the case $k > 0$. We see that the line $y = kx + b$ lies above the x-axis for all x's found to the right of the point x_0, that is, for all x's from the interval $(x_0, +\infty)$, and below the x-axis for all x's found to the left of the point x_0, that is, for all x's from the interval $(-\infty, x_0)$.

Thus, for $k > 0$ Inequality (1) is fulfilled on the interval $(x_0, +\infty)$ and Inequality (2) on the interval $(-\infty, x_0)$.

For $k < 0$, as is seen from Fig. 40, Inequality (1) is fulfilled on the interval $(-\infty, x_0)$, and Inequality (2) on the interval $(x_0, +\infty)$.

Example 1. Using the graphical method, solve the inequalities

$$2x + 1 > 0, \tag{4}$$

$$2x + 1 < 0. \tag{5}$$

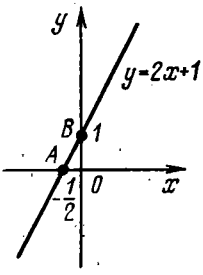


Fig. 41

Let us draw in the Cartesian system Oxy the straight line

$$y = 2x + 1. \tag{6}$$

For this purpose, we must know two of its points. We take the point of intersection of the given line with the x-axis as the first point. It will be useful all the same. Setting $y = 0$ in Formula (6), we get

$$0 = 2x + 1$$

Its solution

$$x = x_0 = -\frac{1}{2}$$

is the abscissa of the point A of intersection of the straight line and the x-axis. Thus, $A\left(-\frac{1}{2}, 0\right)$.

We may take the point B of intersection of the straight line with the y-axis as the second desired point. Its abscissa $x = 0$, and the ordinate

$$y = 2 \cdot 0 + 1 = 1.$$

Thus, $B\ (0, 1)$.

We then draw a straight line through the points A and B. This is just the line $y = 2x + 1$ (Fig. 41).

As is seen from Fig. 41, Inequality (4) is fulfilled on the interval $\left(-\frac{1}{2}, +\infty\right)$ and (5) on the interval $\left(-\infty, -\frac{1}{2}\right)$.

SECTION TEST

QUESTIONS

1. How is it possible to solve first-degree inequalities applying the graphical method?

EXERCISES

1. Applying the graphical method, solve the following inequalities:
(a) $2x + 5 < 0$; (b) $-x - 4 > 0$; (c) $-3x + 2 < 0$.

Sec. 19.3. Linear Inequalities

An inequality whose both sides are polynomials of degree not higher than the first with respect to its variable x or zero is called *linear*.

The following inequalities may serve as examples of linear inequalities:

$$3x + 2 < 0; \quad 5x + 7 < x - 5; \quad -7x + 7 < -7x + 9;$$
$$2x + 7 > 2x + 5; \quad 3x + 2 + x > x - 1 + x.$$

It is clear that any inequality of the first degree is a particular case of a linear inequality. The terms of the polynomials in the left- and right-hand sides of a linear inequality are called the terms of this inequality.

Two inequalities in the unknown x are called *equivalent* if any solution of the first inequality is a solution of the second and, vice versa, any solution of the second inequality is a solution of the first.

When solving inequalities, the following statements (1-4) are used:

1. *The terms of an inequality may be transposed from one side to the other with their signs reversed.*

In other words, if some term of an inequality is transposed from one side to the other with its sign reversed, then an inequality is obtained which is equivalent to the original one.

Thus, for instance, the following inequalities are equivalent

$$2x - 7 < 0, \tag{1}$$
$$2x < 7. \tag{2}$$

Indeed, for any solution x_0 of Inequality (1) the following numerical inequality is true:

$$2x_0 - 7 < 0, \tag{3}$$

from which there follows the validity of the numerical inequality

$$2x_0 < 7. \tag{4}$$

Inequality (4) shows that x_0 is a solution of Inequality (2).

If now x_0 is a solution of Inequality (2), then the numerical inequality (4) is fulfilled, and, consequently, the numerical inequality (3), showing that x_0 is a solution of Inequality (1).

2. *Like terms of an inequality may be collected.*

In other words, if we collect like terms in the left- or right-hand sides of an inequality, then we get an inequality equivalent to the original one.

For instance, the following inequalities are equivalent pairwise:

$$3x - 4 + 5x - 1 > 0 \quad \text{and} \quad 8x - 5 > 0;$$
$$2x + 3 - 1 < x - 2x + 2 \quad \text{and} \quad 2x + 2 < -x + 2.$$

3. *Both sides of an inequality may be multiplied (or divided) by the same positive number without changing the sense of the inequality.*

In other words, if both sides of an inequality are multiplied (or divided) by the same positive number and the sense of the inequality is retained, then an inequality is obtained which is equivalent to the original one.

For instance, the following inequalities are equivalent pairwise:

$$3x+5<0 \text{ and } x+\frac{5}{3}<0;$$
$$2x-3>7x-5 \text{ and } x-\frac{3}{2}>\frac{7}{2}x-\frac{5}{2}.$$

4. *Both sides of an inequality may be multiplied (or divided) by the same negative number, but then the sense of the inequality is reversed* (it is of course forbidden to multiply and divide both sides of an inequality by zero).

In other words, if both sides of an inequality are multiplied (or divided) by the same negative number and the sense of the inequality is reversed, then an inequality is obtained which is equivalent to the original one.

For instance, the following inequalities are equivalent pairwise:

$$7x - 3 > 0 \quad \text{and} \quad -7x + 3 < 0;$$
$$5x + 4 < -3x + 2 \quad \text{and} \quad -5x - 4 > 3x - 2.$$

Formal proof of these statements is carried out just in the same way as in the aforegoing example.

Example 1. Solve the inequality

$$4x - 7 < -2x + 5. \tag{5}$$

Transposing all the terms of Inequality (5) to the left-hand side, we get the inequality

$$4x - 7 + 2x - 5 < 0,$$

which is equivalent to Inequality (5).

Collecting like terms in the left-hand side of the obtained inequality, we get the inequality

$$6x - 12 < 0, \tag{6}$$

which is equivalent to Inequality (5).

Inequality (6) is a first-degree inequality in one unknown x. Let us solve it.

Transposing the constant term of Inequality (6) to the right-hand side and dividing then the obtained inequality by 6, we obtain that the set of all the solutions of Inequality (6) is the set of all $x < 2$ or, which is the same, the set of all the solutions of Inequality (6) forms the interval $(-\infty, 2)$.

Since Inequalities (5) and (6) are equivalent, the set of all the solutions of Inequality (5) also forms the interval $(-\infty, 2)$.

Example 2. Solve the inequality

$$7x + 5 < 7x - 1. \tag{7}$$

Transposing all the terms of Inequality (7) to the left-hand side, we get the inequality

$$7x + 5 - 7x + 1 < 0, \tag{8}$$

which is equivalent to Inequality (7).

Collecting like terms in the left-hand side of Inequality (8), we have

$$0 \cdot x + 6 < 0.$$

Thus, we have obtained a contradiction: the positive number 6 must be less than zero. This contradiction means that there is not a single numerical value of x which would satisfy Inequality (7).

Hence, Inequality (7) has no solution.

Example 3. Solve the inequality

$$9x - 5 > 9x - 6. \tag{9}$$

Transposing all the terms of Inequality (9) to the left-hand side we get the inequality

$$9x - 5 - 9x + 6 > 0, \tag{10}$$

which is equivalent to Inequality (9). Collecting like terms in the left-hand side of Inequality (10), we have

$$0 \cdot x + 1 > 0. \tag{11}$$

We have obtained an inequality true for any values of x. This means that any real number is a solution of Inequality (9), that is, the set of all the solutions of Inequality (9) forms the interval $(-\infty, +\infty)$.

The general recommendation to solving a linear inequality is reduced to the following.

Transpose all the terms of its right-hand side to the left-hand side and collect like terms in the left-hand side. This will result either in a first-degree inequality, or in a true numerical inequality showing that the original inequality has any real number x as its solution, or in a contradictory numerical inequality which shows that the original inequality has no solution.

SECTION TEST

QUESTIONS

1. What is a linear inequality in one unknown?
2. What inequalities are called equivalent?
3. What statements on the equivalence of inequalities do you know?

EXERCISES

1. Solve the following inequalities:
(a) $-x + 3 < 2x + 2 - x$; (b) $2x - 7 > 5x - 4$;
(c) $7x - 3 < 7x + 2$.

Sec. 19.4. Systems of Linear Inequalities in One Unknown

Suppose it is necessary to find all the numerical values of x for which several linear inequalities in one unknown x simultaneously turn into true numerical inequalities. In such cases, we say that it is necessary to solve a system of linear inequalities in one unknown x.

To solve a system of linear inequalities, it is necessary to solve each inequality of this system and then to find the common part of the obtained solutions; it will be the solution of this system.

One inequality is usually placed below the other, the two of them being united with a brace.

Here are examples of systems of linear inequalities in one unknown:

$$\begin{cases} 2x-3<x-2, \\ 3x-4>x, \\ 6x-5>0; \end{cases} \qquad \begin{cases} x-2>0, \\ 2x-3<0; \end{cases} \qquad \begin{cases} x-5>2x+7, \\ 3x-1>2x+4, \\ x-2>0. \end{cases}$$

Consider several examples of solving systems of linear inequalities.

Example 1. Solve the system of inequalities

$$\begin{cases} 2x+4>0, \\ -4x+8<0. \end{cases} \tag{1}$$

Solving the first inequality of the given system, we get that the set of all of its solutions forms the interval $(-2, +\infty)$. Solving the second inequality, we find that the set of all of its solutions forms the interval $(2, +\infty)$. To solve System (1), it is necessary to find the numerical values of x for which both inequalities of System (1) turn simultaneously into true numerical inequalities, that is, it is necessary to find the common part of the intervals $(-2, +\infty)$

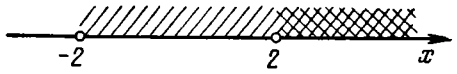


Fig. 42

and $(2, +\infty)$. Draw the coordinate axis x and mark both intervals on it. As is seen from Fig. 42, the common part of these intervals is the interval $(2, +\infty)$.

Consequently, the set of all the solutions of System (1) forms the interval $(2, +\infty)$.

Example 2. Solve the system of inequalities

$$\begin{cases} 5x-23<0, \\ 12x-13>0. \end{cases} \tag{2}$$

Solving each inequality of System (2), we find that the set of all the solutions of the first inequality are all $x<\frac{23}{5}$, and the set of all the solutions of the second inequality are all $x>\frac{13}{12}$.

Fig. 43

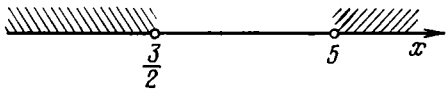


Fig. 44

The set of all the solutions of System (2) will be the set of the numerical values of x for which both inequalities of System (2) turn simultaneously into true numerical inequalities. Consequently, this will be those x's that are greater than $\frac{13}{12}$, but less than $\frac{23}{5}$, that is, all x's from the interval $\frac{13}{12}<x<\frac{23}{5}$ (Fig. 43).

Thus, the set of all the solutions of System (2) forms the interval $\left(\frac{13}{12}, \frac{23}{5}\right)$.

Example 3. Solve the system of inequalities

$$\begin{cases} x-5>0, \\ 2x-3<0. \end{cases} \tag{3}$$

The set of all the solutions of the first inequality of the given system are all $x > 5$, and the set of all the solutions of the second are all $x < \frac{3}{2}$.

The solution of System (3) may include only those x's which are greater than 5, but less than $\frac{3}{2}$. It is clear that there are no such x's (see Fig. 44). Consequently, System (3) has no solutions.

SECTION TEST

QUESTIONS

1. What is called a system of linear inequalities in one unknown?
2. How can a system of linear inequalities be solved?

EXERCISES

1. Solve the following systems of inequalities:

(a) $\begin{cases} 2x+5>0, \\ -x+3>0; \end{cases}$ (b) $\begin{cases} 3x-4<0, \\ x-2>0; \end{cases}$ (c) $\begin{cases} -x+3<0, \\ x+7<0. \end{cases}$

Chapter 20

SECOND-DEGREE INEQUALITIES IN ONE UNKNOWN

Sec. 20.1. The Notion of a Second-degree Inequality in One Unknown

The general inequality of the second degree in one unknown x is

$$ax^2 + bx + c > 0 \tag{1}$$

or

$$ax^2 + bx + c < 0, \tag{2}$$

where a, b, and c are given numbers, and $a \neq 0$.

The number a is called the coefficient of x^2, the number b the coefficient of x, the number c the constant term. The discriminant $D = b^2 - 4ac$ of the quadratic trinomial $ax^2 + bx + c$ is said to be also the discriminant of Inequalities (1) and (2).

The following inequalities may serve as examples of inequalities of the second degree:

$$3x^2 - 4x + 5 > 0; \quad -6x^2 - 2x + 1 < 0,$$
$$-7x^2 + 5x < 0; \quad -9x^2 - 2 > 0.$$

Let us recall that the solution of an inequality in one unknown x is defined as such a number x_0 which, when substituted into the inequality for x turns it into a true numerical inequality, to solve an inequality means to find all of its solutions. Let us also recall that two inequalities in one unknown x are called equivalent to each other if any solution of the first inequality is a solution of the second and, vice versa, any solution of the second inequality is a solution of the first.

When solving second-degree inequalities, we shall use the statements on the equivalence of inequalities given in the preceding chapter. In particular, use will be made of the following statements:

if both sides of an inequality are multiplied (or divided) by a positive number and the sense of the inequality is retained, then a new inequality is obtained which is equivalent to the original one;

if both sides of an inequality are multiplied (or divided) by a negative number and the sense of the inequality is reversed, then a new inequality is obtained which is equivalent to the original one.

Note that if a is a negative number, then, multiplying Inequality (1) by (—1), we get the equivalent inequality

$$(-a)\,x^2 + (-b)\,x + (-c) < 0$$

in which the coefficient of x^2 is already positive.

Similarly, if a is a negative number, then, multiplying Inequality (2) by (—1), we get the equivalent inequality

$$(-a)\,x^2 + (-b)\,x + (-c) > 0$$

in which the coefficient of x^2 is also positive.

Taking this into account, we shall further consider the solutions of Inequalities (1) and (2) supposing that a is a positive number. It will become clear later that the solution of a second-degree inequality depends on the sign of its discriminant. Therefore in the next three sections we shall consider the solution of these inequalities under the following conditions: (a) $D > 0$; (b) $D = 0$; (c) $D < 0$.

SECTION TEST

QUESTIONS

1. What is an inequality of the second degree?
2. What does it mean to solve an inequality in one unknown?
3. What is a solution of an inequality in one unknown?
4. What does it mean that two inequalities are equivalent?

5. What statements on the equivalence of inequalities do you know?

EXERCISES

1. Write five inequalities of the second degree.

2. Is the number $\left(-\frac{3}{7}\right)$ a solution of the following inequalities:

(a) $3x^2 - 4x + 1 > 0$; (b) $7x^2 - 46x - 21 > 0$; (c) $x^2 - 5x + 1 < 0$?

Sec. 20.2. Second-degree Inequalities with a Positive Discriminant

Let it be required to solve the inequality

$$ax^2 + bx + c > 0, \tag{1}$$

where a, b, and c are given numbers, and a and $D = b^2 - 4ac$ are positive.

As we know (see Sec. 17.1), in this case

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \tag{2}$$

where x_1 and x_2 are the roots of the trinomial $ax^2 + bx + c$.

Therefore Inequality (1) may be rewritten as follows:

$$a(x - x_1)(x - x_2) > 0. \tag{3}$$

Since a is positive, the inequality

$$(x - x_1)(x - x_2) > 0 \tag{4}$$

is equivalent to Inequality (3).

Since $D > 0$, we have: $x_1 \neq x_2$. For definiteness, we shall assume that $x_1 < x_2$. Let us draw the coordinate axis x and mark on it the

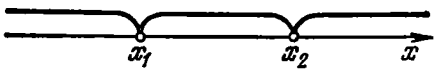


Fig. 45

points x_1 and x_2 (Fig. 45). These points divide the axis x into three parts: $x < x_1$, $x_1 < x < x_2$, $x_2 < x$, in other words, into three intervals:

$$(-\infty, x_1); \quad (x_1, x_2); \quad (x_2, +\infty).$$

Let x belong to the interval $(-\infty, x_1)$, then obviously

$$x - x_1 < 0 \quad \text{and} \quad x - x_2 < 0,$$

and, consequently,

$$(x - x_1)(x - x_2) > 0.$$

Let, further, x belong to the interval (x_1, x_2), then

$$x - x_1 > 0 \quad \text{and} \quad x - x_2 < 0,$$

whence

$$(x - x_1)(x - x_2) < 0.$$

Finally, let x belong to the interval $(x_2, +\infty)$, then

$$x - x_1 > 0 \quad \text{and} \quad x - x_2 > 0,$$

and, consequently,

$$(x - x_1)(x - x_2) > 0.$$

Since inequalities (4) and (1) are equivalent, we have proved that under the condition $D = b^2 - 4ac > 0$ the set of all the solutions x of Inequality (1) consists of two intervals: $(-\infty, x_1)$ and $(x_2, +\infty)$, where x_1 and x_2 $(x_1 < x_2)$ are the roots of the trinomial $ax^2 + bx + c$.

Simultaneously, we have shown that under the indicated condition the set of all the solutions x of the inequality

$$ax^2 + bx + c < 0$$

forms the interval (x_1, x_2).

Example 1. Solve the inequality

$$x^2 - 5x + 6 < 0. \tag{5}$$

We compute the discriminant of Inequality (5):

$$D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1 > 0.$$

Consequently, the quadratic trinomial $x^2 - 5x + 6$ has the roots

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 1}{2},$$

that is, $x_1 = 2$, $x_2 = 3$.

Therefore Inequality (5) may be rewritten as

$$(x - 2)(x - 3) < 0. \tag{6}$$

Let us draw the coordinate axis x and mark on it the points 2 and 3 (Fig. 46).

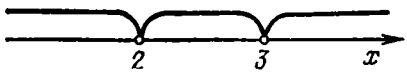


Fig. 46

It is easy to see that the expression

$$(x - 2)(x - 3)$$

is positive for any x situated to the right of the point 3;
negative for any x lying between the points 2 and 3;
positive for any x located to the left of the point 2.

Consequently, the set of all the solutions of Inequality (6), and, hence, of Inequality (5) is represented by the interval (2, 3).

Example 2. Solve the inequality

$$-x^2 - x + 6 > 0. \tag{7}$$

Multiplying Inequality (7) by (—1), we get an equivalent inequality:

$$x^2 + x - 6 < 0. \tag{8}$$

The coefficient of x^2 in Inequality (8) is positive. The discriminant of this inequality

$$D = b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot (-6) = 25 > 0.$$

Consequently, the roots of the quadratic trinomial $x^2 + x - 6$ are

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm 5}{2},$$

that is, $x_1 = -3$, $x_2 = 2$.

Let us draw the coordinate axis x and mark on it the points —3 and 2 (Fig. 47).

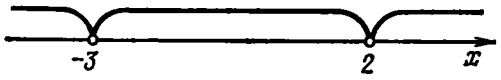


Fig. 47

Reasoning as above, we obtain that the set of all the solutions of Inequality (8) and, hence, also of Inequality (7) forms the interval (—3, 2).

SECTION TEST

QUESTIONS

1. How do we solve second-degree inequalities with a positive discriminant?

EXERCISES

1. Solve the following inequalities:
(a) $-3x^2 + 5x + 2 > 0$; (b) $-x^2 + x + 2 < 0$;
(c) $x^2 - x - 6 > 0$.

Sec. 20.3. Second-degree Inequalities with a Discriminant Equal to Zero

Let it be required to solve the inequality

$$ax^2 + bx + c > 0, \tag{1}$$

where a, b, and c are given numbers, a is positive, and $D = b^2 - 4ac = 0$.

As we know (see Sec. 17.2), in this case

$$ax^2 + bx + c = a(x - x_0)^2, \tag{2}$$

where $x_0 = -\frac{b}{2a}$. Therefore Inequality (1) may be rewritten as follows:

$$a(x - x_0)^2 > 0, \tag{3}$$

where $x_0 = -\frac{b}{2a}$ is the root of the trinomial $ax^2 + bx + c$. For $x = x_0$ the polynomial $(x - x_0)^2$ is equal to zero. And for any numerical value $x \neq x_0$ the polynomial $(x - x_0)^2$ and, consequently, the polynomial $a(x - x_0)^2$, takes on positive values.

Consequently, the solution of Inequality (1) will be represented by any real number x, except for $x = x_0$. In other words, the set of all the solutions of Inequality (1) consists of two intervals: $(-\infty, x_0)$ and $(x_0, +\infty)$, where x_0 is the root of the trinomial $ax^2 + bx + c$.

The aforegoing also implies that the inequality

$$ax^2 + bx + c < 0,$$

where a, b, and c are given numbers, a being positive, and $D = b^2 - 4ac = 0$, has no solution.

Example 1. Solve the inequality

$$4x^2 + 4x + 1 > 0. \tag{4}$$

We find the discriminant of Inequality (4):

$$D = b^2 - 4ac = 4^2 - 4 \cdot 4 \cdot 1 = 0.$$

Then Inequality (4) may be rewritten in the form

$$4\left(x + \frac{1}{2}\right) > 0, \tag{5}$$

whence it is seen that the solution of Inequality (5), and, hence, of Inequality (4) will be represented by any $x \neq -\frac{1}{2}$, that is, the set of all the solutions of Inequality (4) consists of two intervals: $\left(-\infty, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, +\infty\right)$.

Example 2. Solve the inequality

$$x^2 - 6x + 9 < 0. \tag{6}$$

We find the discriminant of Inequality (6):

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 9 \cdot 1 = 0.$$

Hence, Inequality (6) may be rewritten in the form

$$(x - 3)^2 < 0, \tag{7}$$

whence it is obvious that there is not a single x satisfying Inequality (7). Hence, Inequality (6) has no solution.

SECTION TEST

QUESTIONS

1. Does a second-degree inequality with a zero discriminant always have a solution?

EXERCISES

1. Solve the following inequalities:
(a) $25x^2 - 40x + 16 > 0$; (b) $-x^2 + 4x - 4 > 0$;
(c) $4x^2 - 12x + 9 < 0$.

Sec. 20.4. Second-degree Inequalities with a Negative Discriminant

Consider the problem on solving the inequalities

$$ax^2 + bx + c > 0, \tag{1}$$

$$ax^2 + bx + c < 0, \tag{2}$$

where a, b, c are given numbers, a being positive, and the discriminant $D = b^2 - 4ac$ is negative.

Let us begin with examples.

Example 1. Solve the inequality

$$5x^2 - 6x + 2 > 0. \tag{3}$$

The discriminant of this inequality

$$D = b^2 - 4ac = 36 - 4 \cdot 5 \cdot 2 = -4 < 0.$$

Let us transform the quadratic trinomial in the left-hand side of (3) by applying the method of isolating a perfect square:

$$5x^2 - 6x + 2 = 5\left(x^2 - 2x \cdot \frac{3}{5} + \left(\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2 + \frac{2}{5}\right)$$

$$= 5\left[\left(x - \frac{3}{5}\right)^2 + \frac{1}{25}\right].$$

Thus, Inequality (3) may be written in the form

$$5\left[\left(x-\frac{3}{5}\right)^2+\frac{1}{25}\right]>0,$$

whence it is seen that this inequality is fulfilled for any real x or, in other words, on the interval $(-\infty, +\infty)$.

Example 2. Solve the inequality

$$3x^2+x+1<0. \tag{4}$$

The discriminant of Inequality (4)

$$D=b^2-4ac=1-4\cdot 3\cdot 1<0.$$

Separating a perfect square in the left-hand side of (4), we get

$$3x^2+x+1=3\left(x^2+\frac{x}{3}+\frac{1}{3}\right)$$

$$=3\left(x^2+2\cdot\frac{x}{6}+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2+\frac{1}{3}\right)=3\left[\left(x+\frac{1}{6}\right)^2+\frac{11}{36}\right].$$

Consequently, Inequality (4) may be written in the following way:

$$3\left[\left(x+\frac{1}{6}\right)^2+\frac{11}{36}\right]<0.$$

The left-hand side of the obtained inequality is positive for any x, therefore this inequality cannot be fulfilled for any x.

Thus, Inequality (4) has no solution.

The general form of the solution of Inequalities (1) and (2) can be obtained by generalizing the considerations carried out in the above examples.

Separating a perfect square from the trinomial ax^2+bx+c $(a>0)$, we get

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{-D}}{2a}\right)^2\right]. \tag{5}$$

Here, it should be taken into account that we supposed that $D<0$, therefore $(-D)>0$, and the square root $\sqrt{-D}$ has sense.

From Equality (5) it follows that Inequality (1) may be rewritten in the following way:

$$a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{-D}}{2a}\right)^2\right]>0,$$

whence, taking into consideration that $a>0$, it is seen that Inequality (1) is fulfilled for all real x's, in other words, on the interval $(-\infty, +\infty)$.

Hence, obviously, it also follows that under the above conditions (that is, $a>0$, $D<0$) Inequality (2) $ax^2+bx+c<0$ has no solution.

SECTION TEST

QUESTIONS

1. Does a second-degree inequality with a negative discriminant always have a solution?

EXERCISES

1. Solve the following inequalities:
(a) $4x^2 + x + 3 < 0$; (b) $-4x^2 + 5x - 2 < 0$;
(c) $3x^2 - 13x + 30 > 0$.

Sec. 20.5. Graphs Applied to Solving Second-degree Inequalities

Consider the graphical method of solving the inequalities

$$ax^2 + bx + c > 0, \tag{1}$$

$$ax^2 + bx + c < 0. \tag{2}$$

We shall suppose that a is positive. In the Cartesian coordinate system Oxy, consider the graph (Fig. 48) of the quadratic trinomial

$$y = ax^2 + bx + c.$$

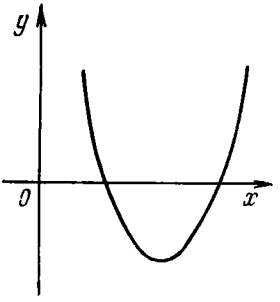


Fig. 48

As we know, this graph is represented by the parabola $y = ax^2$ translated to a certain place of the plane Oxy. Obviously, for the values of x for which the given parabola is situated above the x-axis Inequality (1) is fulfilled, Inequality (2) being fulfilled for the values of x for which it is located below the x-axis.

Let us pass to the examples.

Example 1. Solve the inequalities

$$x^2 + x + 1 > 0 \tag{3}$$

and

$$x^2 + x + 1 < 0. \tag{4}$$

Using the method of isolating a perfect square, we get

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Consequently, the quadratic trinomial

$$y = x^2 + x + 1 \tag{5}$$

may be written in the form

$$y=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

But then, as we know, the graph of the trinomial (5) in the Cartesian coordinate system Oxy represents the parabola $y = x^2$ translated so that its vertex is found at the point $A\left(-\frac{1}{2}, \frac{3}{4}\right)$ (Fig. 49).

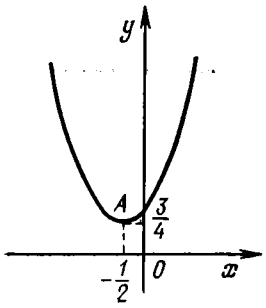


Fig. 49

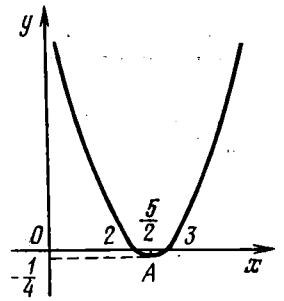


Fig 50

The given parabola is entirely situated above the x-axis, that is, $y > 0$ for all x's or $x^2 + x + 1 > 0$ for all x's. We may also say that this inequality is fulfilled on the interval $(-\infty, +\infty)$.

It is also obvious that there is no such x for which Inequality (4) would be fulfilled.

Thus, we have solved our problem. Any real number x is a solution of Inequality (3), while Inequality (4) has no solution.

Example 2. Solve the inequalities

$$x^2 - 5x + 6 > 0 \tag{6}$$

and

$$x^2 - 5x + 6 < 0 \tag{7}$$

We have $x^2-5x+6=\left(x-\frac{5}{2}\right)^2-\frac{1}{4}$

Consequently, the quadratic trinomial $y = x^2 - 5x + 6$ may be written in the form

$$y=\left(x-\frac{5}{2}\right)^2-\frac{1}{4}.$$

But then the graph of this trinomial is the parabola $y = x^2$ translated so that its vertex is found at the point $A\left(\frac{5}{2}, -\frac{1}{4}\right)$ (Fig. 50)

The parabola opens upward, intersecting the x-axis at two points; its vertex A is situated below the x-axis.

The points of intersection with the x-axis are of great importance. They have the ordinate $y = 0$, their abscissas being the roots of the equation

$$0 = x^2 - 5x + 6,$$

that is, $x_1 = 2$, $x_2 = 3$.

Let us mark these points on the x-axis. It is seen from Fig. 50 that our parabola is located below the x-axis on the interval (2, 3) and above the x-axis on the intervals $(-\infty, 2)$, $(3, \infty)$.

Hence it follows that all the solutions of Inequality (7) fill up the interval (2, 3), and all the solutions of Inequality (6) fill up two intervals: $(-\infty, 2)$ and $(3, +\infty)$.

Example 3. Solve the inequalities

$$-x^2 + 2x - 1 > 0 \tag{8}$$

and

$$-x^2 + 2x - 1 < 0. \tag{9}$$

Multiplying each of these inequalities by (-1), we get the inequalities

$$x^2 - 2x + 1 < 0 \tag{10}$$

and

$$x^2 - 2x + 1 > 0, \tag{11}$$

which are respectively equivalent to (8) and (9).

We have done this to take advantage of the graph of the quadratic trinomial—the parabola opening upward.

Further we have

$$x^2 - 2x + 1 = (x - 1)^2.$$

Therefore the quadratic trinomial $y = x^2 - 2x + 1$ may be written in the form $y = (x - 1)^2$. Its graph is a parabola shown in Fig. 51. It opens upward, its vertex lying on the x-axis.

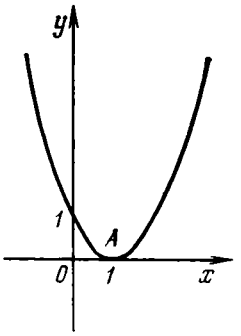


Fig. 51

Inequality (10) has no solution since all the points of the given parabola have non-negative ordinates y. And Inequality (11) is fulfilled for all x's except $x = 1$.

Accordingly, Inequality (8) has no solution, while the set of all the solutions of Inequality (9) consists of two intervals: $(-\infty, -1)$ and $(1, +\infty)$.

SECTION TEST

QUESTIONS

1. How is it possible to solve second-degree inequalities with the aid of their graphs?

EXERCISES

1. Applying graphs, solve the following inequalities:
(a) $-x^2 + 4x - 4 < 0$; (b) $4x^2 + x + 3 > 0$;
(c) $3x^2 - 5x - 2 > 0$.

Chapter 21

RATIONAL INEQUALITIES

Sec. 21.1. The Method of Intervals

Draw the coordinate axis x and mark a point x_0 on it (Fig. 52). As was already noted in Chapter 19, the point x_0 divides the x-axis into two parts:

(1) for any x situated to the right of x_0 the binomial $x - x_0$ is positive;

(2) for any x situated to the left of x_0 the binomial $x - x_0$ is negative.

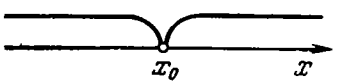


Fig. 52

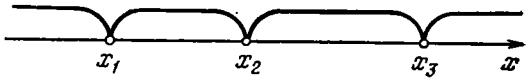


Fig. 53

This property underlies the method of intervals. Let, for instance, it be required to solve the inequality

$$(x - x_1)(x - x_2)(x - x_3) > 0 \tag{1}$$

or the inequality

$$(x - x_1)(x - x_2)(x - x_3) < 0, \tag{2}$$

where the numbers x_1, x_2, x_3 are different and arranged in the increasing order:

$$x_1 < x_2 < x_3.$$

Mark the points x_1, x_2, x_3 on the x-axis (Fig. 53). They divide this axis into four intervals: $(-\infty, x_1)$, (x_1, x_2), (x_2, x_3), $(x_3, +\infty)$. Consider the expression

$$A(x) = (x - x_1)(x - x_2)(x - x_3). \tag{3}$$

On the basis of the above mentioned property of the binomial, it is obvious that for any x found to the right of x_3 any binomial in Product (3) is positive since the point x is situated to the right of all the points x_1, x_2, x_3. Therefore $A(x) > 0$ for any x belonging to the interval $(x_3, +\infty)$.

For any x lying between x_2 and x_3 the last factor in Product (3) is negative since x is to the left of the point x_3, and any of the remaining factors is positive since x lies to the right of the points x_1 and x_2. Therefore $A(x) < 0$ for any x from the interval (x_2, x_3).

Analogously, $A(x) > 0$ for any x from the interval (x_1, x_2) and $A(x) < 0$ for any x from the interval $(-\infty, x_1)$.

This reasoning is the basis of the *method of intervals* consisting in the following: mark the points x_1, x_2, x_3 on the x-axis, put the plus sign above the interval $(x_3, +\infty)$, the minus sign above the interval

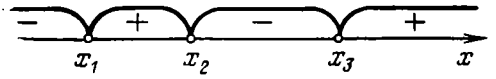


Fig. 54

(x_2, x_3), the plus sign above the interval (x_1, x_2), and the minus sign above the interval $(-\infty, x_1)$ (see Fig. 54). Then the set of all the solutions of Inequality (1) will consist of all the intervals with the plus sign, while the set of all the solutions of Inequality (2) will consist of all the intervals with the minus sign.

Using this method, we can solve the inequalities

$$(x - x_1)(x - x_2)\ldots(x - x_n) > 0 \text{ and } (x - x_1)(x - x_2)\ldots(x - x_n) < 0,$$

where $x_1 < x_2 < \ldots < x_n$ and n is any natural number.

Note that earlier we actually applied this method to solving second-degree inequalities with a positive discriminant.

Example 1. Solve the inequality

$$(x - 1)(x - 2)(x - 3) > 0. \tag{4}$$

We shall solve this inequality using the methods of intervals. Mark the points 1, 2, 3 on the x-axis and above the intervals $(-\infty, 1)$, $(1, 2)$, $(2, 3)$, $(3, +\infty)$ put the plus and minus signs from right to left (Fig. 55).

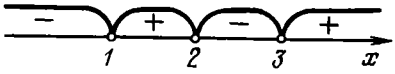


Fig. 55

From Fig. 55 it is seen that the set of all the solutions of Inequality (4) consists of two intervals: $(1, 2)$ and $(3, +\infty)$.

Example 2. Solve the inequality

$$(2 - x)(x^2 - 4x + 3)(x + 1) > 0. \tag{5}$$

Factoring $x^2 - 4x + 3$ into factors

$$x^2 - 4x + 3 = (x - 1)(x - 3),$$

we obtain that Inequality (5) may be written in the form

$$-[x - (-1)](x - 2)(x - 1)(x - 3) > 0. \tag{6}$$

It is clear that Inequality (6) is equivalent to the inequality

$$[x - (-1)](x - 1)(x - 2)(x - 3) < 0. \tag{7}$$

Therefore it remains to solve Inequality (7). It is already written in the form useful for the method of intervals. Mark the points (—1), 1, 2 and 3 on the x-axis (Fig. 56).

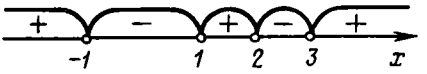


Fig. 56

Then, applying the method of intervals, we find that the set of all the solutions of Inequality (5) consists of two intervals: (—1, 1) and (2, 3).

SECTION TEST

QUESTIONS

1. What is the essence of the method of intervals?

EXERCISES

1. Solve the following inequalities:
(a) $(x - 3)(x^2 - 3x + 2) > 0$; (b) $(x^2 - 1)(x - 2)(x + 3) < 0$;
(c) $(x^2 - 3x - 4)(x^2 + x - 2) < 0$.

Sec. 21.2. Solving Rational Inequalities

Let there be given an algebraic fraction $\frac{A(x)}{B(x)}$, where $A(x)$ and $B(x)$ are polynomials with respect to x. The inequality

$$\frac{A(x)}{B(x)} > 0 \tag{1}$$

is called the *rational inequality*.

The solution of Inequality (1) is defined as such numerical value $x = x_0$ for which Inequality (1) turns into a true numerical inequality.

To solve Inequality (1) means to find all of its solutions.

It is easy to see that any solution of Inequality (1) will be a solution of the inequality

$$A(x) \cdot B(x) > 0, \tag{2}$$

and any solution of Inequality (2) will be a solution of Inequality (1). Hence, Inequalities (1) and (2) are equivalent.

We shall consider only the case when the polynomials $A(x)$ and $B(x)$ are factored into the product of binomials of the form $(x - x_0)$. Then Inequality (2) can be solved using the method of intervals.

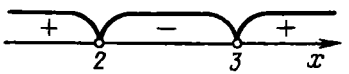


Fig. 57

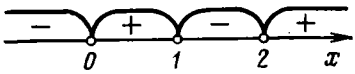


Fig. 58

But Inequalities (1) and (2) are equivalent, therefore there is no need to pass to Inequality (2) since it suffices to apply the method of intervals to Inequality (1) at once.

Example 1. Solve the inequality

$$\frac{x-3}{x-2} > 0. \tag{3}$$

Applying the method of intervals (Fig. 57), we find that the set of all the solutions of Inequality (3) consists of two intervals $(-\infty, 2)$ and $(3, +\infty)$.

Example 2. Solve the inequality

$$\frac{x-1}{x^2-2x} < 0. \tag{4}$$

Factoring the polynomial $x^2 - 2x$, we rewrite Inequality (4) in the form

$$\frac{x-1}{(x-0)(x-2)} < 0. \tag{5}$$

Applying the method of intervals (Fig. 58), we find that the set of all the solutions of Inequality (5) and, hence, also of Inequality (4) consists of two intervals: $(-\infty, 0)$ and $(1, 2)$.

Example 3. Solve the inequality

$$\frac{x^2-2x-3}{x^2-3x+2} < 0. \tag{6}$$

Let us factor into binomial factors the quadratic trinomials

$$x^2 - 2x - 3 \tag{7}$$

and

$$x^2 - 3x + 2. \tag{8}$$

The discriminant of the quadratic trinomial (7)

$$D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-3) = 16 > 0.$$

Therefore the quadratic trinomial (7) has two roots:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 4}{2},$$

that is, $x_1 = -1$, $x_2 = 3$, and is factored into the following factors:

$$x^2 - 2x - 3 = [x - (-1)](x - 3).$$

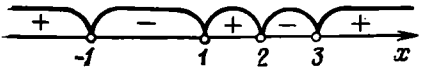


Fig. 59

The discriminant of the quadratic trinomial (8)

$$D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 2 = 1 > 0.$$

Therefore the quadratic trinomial (8) also has two roots:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm 1}{2},$$

that is, $x_1 = 1$, $x_2 = 2$, and is factored into the following factors:

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

Consequently, Inequality (6) may be rewritten in the form

$$\frac{[x-(-1)](x-3)}{(x-1)(x-2)} < 0.$$

Applying the method of intervals (Fig. 59), we find that the set of all the solutions of Inequality (6) consists of two intervals: $(-1, 1)$ and $(2, 3)$.

SECTION TEST

QUESTIONS

1. What is a rational inequality?
2. How can a rational inequality be solved?

EXERCISES

1. Solve the following inequalities:

(a) $\frac{x^2-x-2}{5-x} > 0$, (b) $\frac{2-x}{4+3x-x^2} < 0$, (c) $\frac{x^2-5x+6}{2-x^2-x} > 0$.

Sec. 21.3. Systems of Rational Inequalities

Let there be required to find the numerical values of x for which several rational inequalities turn into true numerical inequalities simultaneously. In such cases we say that it is required to solve a system of rational inequalities in one unknown x.

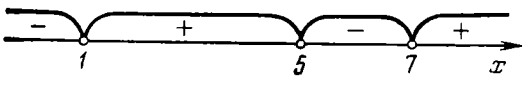


Fig. 60

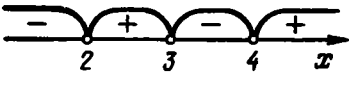


Fig. 61

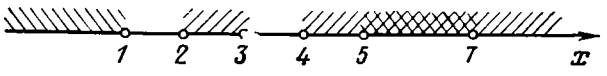


Fig. 62

To solve a system of rational inequalities, we have to find all the solutions of each inequality of the system. Then the common part of the found solutions will just be the solution of the system.

Example 1. Solve the system of inequalities

$$\begin{cases} (x-1)(x-5)(x-7)<0, \\ \dfrac{(x-2)(x-3)}{x-4}>0 \end{cases} \tag{1}$$

We first solve the inequality

$$(x-1)(x-5)(x-7)<0. \tag{2}$$

Applying the method of intervals (Fig. 60), we find that the set of all the solutions of Inequality (2) consists of two intervals: $(-\infty, 1)$ and $(5, 7)$.

Let us now solve the inequality

$$\frac{(x-2)(x-3)}{x-4}>0. \tag{3}$$

Using the method of intervals (Fig. 61), we find that the set of all the solutions of Inequality (3) also consists of two intervals: $(2, 3)$ and $(4, +\infty)$.

Now, we have to find the common part of the solutions of Inequalities (2) and (3). We draw the coordinate axis x and mark on it the found solutions. Now it is clear that the common part of the solutions of Inequalities (2) and (3) is the interval $(5, 7)$ (Fig. 62).

Consequently, the set of all the solutions of the system of inequalities (1) is represented by the interval $(5, 7)$.

Example 2. Solve the system of inequalities

$$\begin{cases} x^2-6x+10<0, \\ \dfrac{x^9-x^3+x+2}{x^4-x^2+1}>0. \end{cases} \tag{4}$$

We first solve the inequality

$$x^2-6x+10<0. \tag{5}$$

Applying the method of separating a perfect square, we may write

$$x^2-6x+10=x^2-2\cdot x\cdot 3+3^2-3^2+10=(x-3)^2+1.$$

Therefore, Inequality (5) may be written in the form

$$(x-3)^2+1<0,$$

whence it is seen that it has no solution.

Now we need not solve the inequality

$$\frac{x^9-x^3+x+2}{x^4-x^2+1}>0,$$

since the answer is already clear: System (4) has no solution.

SECTION TEST

QUESTIONS

1. What is a system of rational inequalities?
2. How do we solve systems of rational inequalities?

EXERCISES

1. Solve the following systems of inequalities:

(a) $\begin{cases} \dfrac{x^2+x}{x^2+5x+6}>0, \\ \dfrac{x+5}{x-1}<0; \end{cases}$ (b) $\begin{cases} x^2+x+1<0, \\ \dfrac{x^2-5}{x+9}>0; \end{cases}$ (c) $\begin{cases} x^2+2x+3>0, \\ \dfrac{x^2-4}{x^2-1}>0. \end{cases}$

Sec. 21.4. Nonstrict Rational Inequalities

Consider the solution of the nonstrict inequalities

$$\frac{A(x)}{B(x)}\geqslant 0, \tag{1}$$

$$\frac{A(x)}{B(x)}\leqslant 0, \tag{2}$$

where $A(x)$ and $B(x)$ are polynomials with respect to x.

If some number x_0 is a solution of Inequality (1), then the numerical inequality $\frac{A(x_0)}{B(x_0)}\geqslant 0$ holds true. Then, by virtue of

determining the sign of a nonstrict inequality, there holds true either the numerical inequality $\frac{A(x_0)}{B(x_0)} > 0$ or the numerical equality $\frac{A(x_0)}{B(x_0)} = 0$. In other words, if the number x_0 is a solution of Inequality (1), then it is a solution either of the inequality

$$\frac{A(x)}{B(x)} > 0, \tag{3}$$

or of the equation

$$\frac{A(x)}{B(x)} = 0. \tag{4}$$

It is also easy to see that any solution of Inequality (3) or any solution of Inequality (4) are solutions of Inequality (1).

Consequently, the set of all the solutions of Inequality (1) is the combination of the set of all the solutions of Inequality (3) and the set of all the solutions of Equations (4).

Analogously, the set of all the solutions of Inequality (2) is the combination of the set of all the solutions of the inequality

$$\frac{A(x)}{B(x)} < 0$$

and the set of all the solutions of the equation

$$\frac{A(x)}{B(x)} = 0.$$

Example 1. Solve the inequality

$$3x - 7 \geqslant 0. \tag{5}$$

We first solve the equation

$$3x - 7 = 0. \tag{6}$$

Its only solution is $x_0 = \frac{7}{3}$. Then, we solve the inequality

$$3x - 7 > 0. \tag{7}$$

Transposing the constant term to the right-hand side and dividing the obtained inequality by 3, we get that the set of all the solutions of Inequality (7) consists of all $x > \frac{7}{3}$. Uniting the set of all the solutions of Inequality (7) and Equation (6), we obtain that the set of all the solutions of Inequality (5) is represented by the half-interval $\left[\frac{7}{3}, +\infty\right)$.

Example 2. Solve the inequality

$$2x^2 - x - 1 \leqslant 0. \tag{8}$$

We first solve the equation

$$2x^2 - x - 1 = 0. \tag{9}$$

Its discriminant

$$D = b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot (-1) = 9 > 0.$$

Consequently, the roots of Equation (9) are

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{4},$$

that is, $x_1 = -\frac{1}{2}$, $x_2 = 1$.

We now solve the inequality

$$2x^2 - x - 1 < 0. \tag{10}$$

Since the quadratic trinomial $2x^2 - x - 1$ has the roots $x_1 = -\frac{1}{2}$ and $x_2 = 1$, Inequality (10) may be written in the form

$$2\left[x - \left(-\frac{1}{2}\right)\right](x-1) < 0.$$

Let us solve it using the method of intervals. Mark the points $\left(-\frac{1}{2}\right)$ and 1 on the x-axis (Fig. 63). Then, taking into consideration

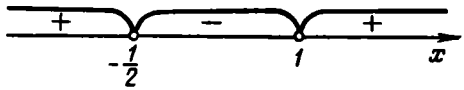


Fig. 63

that $a = 2 > 0$, we put the signs + and −. Now we obtain that the set of all the solutions of Inequality (10) is represented by the interval $\left(-\frac{1}{2}, 1\right)$.

Combining the solutions of Inequality (10) and Equation (9), we get that the set of all the solutions of Inequality (8), is the interval $\left[-\frac{1}{2}, 1\right]$.

Example 3. Solve the inequality

$$9x^2 - 6x + 1 \leqslant 0. \tag{11}$$

First, we solve the equation

$$9x^2 - 6x + 1 = 0. \tag{12}$$

Its discriminant

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 9 \cdot 1 = 0.$$

Consequently, Equation (12) has the only root

$$x_0 = -\frac{b}{2a} = \frac{6}{18} = \frac{1}{3}.$$

We now solve the inequality

$$9x^2 - 6x + 1 < 0. \tag{13}$$

Since the quadratic trinomial $9x^2 - 6x + 1$ has the only root $x_0 = \frac{1}{3}$, Inequality (13) may be written in the form

$$9\left(x - \frac{1}{3}\right)^2 < 0,$$

whence it is seen that there is not a single real number x satisfying this inequality (since the square of any real number cannot be negative). Consequently, Inequality (13) has no solution.

Hence, Inequality (11) has the only solution $x_0 = \frac{1}{3}$.

Example 4. Solve the inequality

$$\frac{(x+2)(x-4)}{(x+3)x} \leqslant 0. \tag{14}$$

We first solve the equation

$$\frac{(x+2)(x-4)}{(x+3)x} = 0. \tag{15}$$

It is easy to see that it has two roots: $x_1 = -2$ and $x_2 = 4$ and no other roots.

Let us now solve the inequality

$$\frac{(x+2)(x-4)}{(x+3)(x-0)} < 0. \tag{16}$$

Applying the method of intervals (Fig. 64), we find that the set of all the solutions of Inequality (16) is contained in two intervals: $(-3, -2)$ and $(0, 4)$.

Fig. 64

Combining the solutions of Equation (15) and Inequality (16), we get that the set of all the solutions of Inequality (14) is represented by two half-intervals: $(-3, -2]$ and $(0, 4]$.

SECTION TEST

QUESTIONS

1. How may nonstrict inequalities be solved?

EXERCISES

1. Solve the following inequalities:

(a) $-2x-5 \leqslant 0$; (b) $x^2-4x-21 \geqslant 0$; (c) $\frac{(x+4)(x+1)}{x-5} \geqslant 0$.

Chapter 22

POWER FUNCTIONS

Sec. 22.1. Some Properties of Natural Powers

We shall suppose that a is a real number and n a natural number. From the definition of the nth power of a number it follows that

$a^1 = a$; $a^2 = a \cdot a$; $a^3 = a^2 \cdot a$; $a^4 = a^3 \cdot a$; . . . $a^n = a^{n-1} \cdot a$.

The following properties of natural powers take place:

(1) $0^n = 0$;
(2) $1^n = 1$;
(3) if $0 < a$, then $0 < a^n$;
(4) if $0 < a < b$, then $a^n < b^n$.

Let us prove these statements. But these proofs are actually suitable for small natural n.

(1) $0^2 = 0 \cdot 0 = 0$;
$0^3 = 0^2 \cdot 0 = 0 \cdot 0 = 0$;
$0^4 = 0^3 \cdot 0 = 0 \cdot 0 = 0$;
. . . . . .

Applying these considerations $n - 1$ times we get $0^n = 0$.

(2) $1^2 = 1 \cdot 1 = 1$;
$1^3 = 1^2 \cdot 1 = 1 \cdot 1 = 1$;
$1^4 = 1^3 \cdot 1 = 1 \cdot 1 = 1$;
. . . . . .

Applying these considerations $n - 1$ times, we get $1^n = 1$.

(3) Let $0 < a$; thus, a is a positive number. Multiplying the inequality $0 < a$ by a, we get $a \cdot 0 < a^2$ or $0 < a^2$. Multiplying this inequality by a, we get

$$0 \cdot a < a^3, \text{ or } 0 < a^3.$$

Multiplying this inequality by a, we get

$$0 \cdot a < a^4, \text{ or } 0 < a^4 \text{ and so on.}$$

Applying these considerations $n - 1$ times, we get

$$0 < a^n.$$

(4) Let $0 < a < b$, that is $0 < a$ and $a < b$; then $b > 0$. Multiplying the inequality $a < b$ by a and by b, we get

$$a^2 < ab \text{ and } ab < b^2,$$

whence

$$a^2 < b^2.$$

Multiplying now the inequality $a < b$ by a^2 and the inequality $a^2 < b^2$ by b, we get

$$a^3 < a^2b \text{ and } a^2b < b^3,$$

whence

$$a^3 < b^3.$$

Multiplying the inequality $a < b$ by a^3 and the inequality $a^3 < b^3$ by b, we get

$$a^4 < a^3b \text{ and } a^3b < b^4,$$

whence

$$a^4 < b^4 \text{ and so on.}$$

Applying these considerations $n - 1$ times, we get

$$a^n < b^n.$$

Remark. The above considerations seem to be irreproachable, but in fact they can be realized only for small n. And for any n there arises a necessity to use another method of proof in which actual fulfillment of all mentioned stages of reasoning is not required. This method of proof is based on the principle of complete induction.

Consider several examples illustrating how the properties (1) - (4) are applied.

Example 1. If $0 \leqslant a < b$, then $a^n < b^n$.

Indeed, for $a = 0$ we have $0 < b$, and then, by Property (3), $0 < b^n$, and if $0 < a$, then $a^n < b^n$, by virtue of Property (4).

Example 2. If $0 \leqslant a < 1$, then $a^n < 1$.

Indeed, the inequalities $0 \leqslant a < 1$ imply that $a^n < 1^n$, but $1^n = 1$, and therefore $a^n < 1$.

Example 3. If $1 < b$, then $1 < b^n$. Indeed, taking into account that $0 < 1$, that is, that $0 < 1 < b$, we get $1^n < b^n$. Whence, bearing in mind that $1^n = 1$, we get $1 < b^n$.

SECTION TEST

EXERCISES

1. Prove that:
(a) if $0 \leqslant a \leqslant b$, then $a^n \leqslant b^n$; (b) if $0 < a \leqslant b$, then $a^n \leqslant b^n$;
(c) if $0 \leqslant a$, then $0 \leqslant a^n$.
2. Formulate the properties of natural powers.

Sec. 22.2. Principle of Complete Induction

Prior to formulating the principle of complete induction in general form, we are going to clarify it by the following example. It is required to prove that if $a > 0$, then

$$0 < a^n \tag{1}$$

for any natural n.

According to the principle of complete induction, for Inequality (1) to be regarded true for all natural n, it suffices to check whether the following statements are fulfilled:

(1) Inequality (1) holds true for $n = 1$;

(2) if it is assumed that for a certain $n = k$ Inequality (1) is true, that is, if the inequality $0 < a^k$ takes place, then it is also true for $n = k + 1$, that is, the following inequality takes place:

$$0 < a^{k+1}.$$

Statement (1) is actually fulfilled, since setting in Inequality (1) $n = 1$, we get the inequality $0 < a$ which is true by hypothesis. Statement (2) is also fulfilled: in fact, if we suppose that the inequality $0 < a^k$ is true, then, on multiplying it by the positive number a, we get the true inequality

$$0 < a^{k+1}.$$

Thus, the statements (1) and (2) are fulfilled. But then the principle of complete induction states that Inequality (1) holds true for any natural n.

Let us now formulate the principle of complete induction.

If a property depending on a natural n, firstly, is true for $n = 1$ and, secondly, from the assumption that it is true for $n = k$ it follows that it is true for $n = k + 1$, then it is regarded that this property is true for any natural n.

A proof based on the principle of induction is called the proof by induction or the proof by the method of induction.

Note that not only proofs, but also definitions of the nth power, strictly speaking, should be given by induction.

For example, we say that a^n for natural n is such a number which is defined in the following way:

$$a^1 = a \text{ and } a^{k+1} = a^k \cdot a \tag{2}$$

for any natural k.

Using this definition, we get, for instance, that

$$a^5 = a^4 \cdot a = a^3 \cdot a \cdot a = a^2 \cdot a \cdot a \cdot a = a^1 \cdot a \cdot a \cdot a \cdot a = aaaaa.$$

Example 1. The equality

$$0^n = 0 \tag{3}$$

for any natural n is proved by induction in the following way. For $n = 1$ Equality (3) is trivial. If we take for granted the equality $0^k = 0$, then it will follow that $0^{k+1} = 0^k \cdot 0 = 0 \cdot 0 = 0$. Consequently, according to the principle of complete induction, Equality (3) should be regarded as true for any natural n.

Example 2. If $0 < a < b$, then

$$a^n < b^n \tag{4}$$

for any natural n.

This statement is proved by induction as follows.

Setting $n = 1$ in Inequality (4), we get a true inequality $a < b$ which is given by hypothesis. Let then the following inequality be true:

$$a^k < b^k. \tag{5}$$

Since, by hypothesis, a is a positive number, a^k is also a positive number—this was proved above. Multiplying the inequality $a < b$ by a^k and Inequality (5) by b, we get

$$a^{k+1} < a^k b \text{ and } a^k b < b^{k+1},$$

whence

$$a^{k+1} < b^{k+1}.$$

Consequently, according to the principle of complete induction, Inequality (5) should be regarded as true for any natural n.

Example 3. Prove by induction the inequality

$$(1 + b)^n \geqslant 1 + nb \qquad (b > 0), \tag{6}$$

where n is a natural number.

Indeed, since

$$(1 + b)^1 = 1 + b,$$

we have thus proved (6) for $n = 1$.

Suppose now that Property (6) is true for $n = k$, that is, that the inequality

$$(1 + b)^k \geqslant 1 + kb$$

is true for some natural k. Then

$$(1+b)^{k+1} = (1+b)^k (1+b) \geqslant (1+kb)(1+b)$$
$$= 1 + (k+1)b + kb^2 > 1 + (k+1)b,$$

that is, we have proved (6) for $n = k + 1$. But then (6) is true for any natural n.

Example 4. The inequality

$$a^{m+n} = a^m a^n, \tag{7}$$

where m and n are any natural numbers, is proved by induction in the following way.

Let us assign an arbitrary natural m and carry out induction with respect to n. For $n = 1$ Equality (7) is true (see (2)), by the definition of power:

$$a^{m+1} = a^m \cdot a = a^m \cdot a^1.$$

Let now the following equality be true:

$$a^{m+k} = a^m a^k.$$

Then, by the definition of power (see (2)),

$$a^{m+k+1} = a^{m+k} \cdot a^1 = a^m a^k a^1 = a^m a^{k+1},$$

that is,

$$a^{m+k+1} = a^m a^{k+1}.$$

But then, according to the principle of complete induction, Equality (7) takes place for any natural n with an arbitrarily taken natural m.

This shows that Equality (7) is true for any natural n and m.

SECTION TEST

QUESTIONS

1. What does the principle of complete induction consist in?
2. What is the proof by induction?

EXERCISES

1. Prove by induction the following equalities for natural n and m:
(a) $1^n = 1$; (b) $a^n b^n = (ab)^n$; (c) $(a^m)^n = a^{mn}$.
2. Let $a < 0$. Prove by induction that $a^m > 0$ for an even m and $a^m < 0$ for an odd m.

Sec. 22.3. Properties of Functions $y = x^n$

The functions

$$y = x^2; \quad y = x^3; \quad y = x^4; \quad y = x^5; \quad \ldots \qquad (1)$$

are called *power* functions.

In the general case, we may consider the power function

$$y = x^n, \qquad (2)$$

where n is an arbitrary natural number exceeding unity.

The first of these functions

$$y = x^2$$

was studied earlier. Its graph was named the parabola. Now we shall call this graph a parabola of the second degree.

Let us recall that the graph of the function $y = x^2$ is defined as the totality (set) of points $A\ (x, y)$ of the coordinate plane Oxy whose coordinates satisfy the condition $y = x^2$. Similarly, the graph of the function $y = x^3$ is defined as the totality of points $A\ (x, y)$ whose coordinates satisfy the condition $y = x^3$.

The graph of the function $y = x^n$ is defined as the totality of points $A\ (x, y)$ whose coordinates satisfy the condition $y = x^n$.

The graph of the function $y = x^3$ is a parabola of the third degree, the graph of the function $y = x^4$ a parabola of the fourth degree, and so forth.

In general, the graph of the function

$$y = x^n,$$

where n is a natural number and $n \geqslant 2$ is called a *parabola of degree n*.

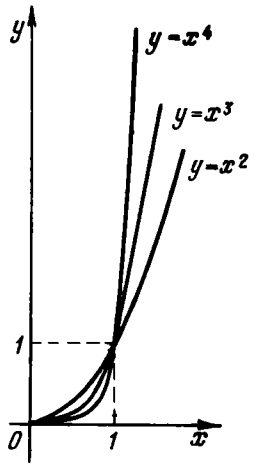


Fig. 65

Figure 65 represents the graphs of the functions

$$y = x^2; \quad y = x^3; \quad y = x^4$$

drawn in the same Cartesian coordinate system for the time being for nonnegative values of $x\ (x \geqslant 0)$. These are continuous curves passing through the points (0, 0) and (1, 1).

The property of continuity requires a formal proof. Below, we shall carry out such a proof for the function $y = x^3$.

The fact that these curves pass through the points (0, 0) and (1, 1) is obvious. Indeed, if in Formulas (1) and (2) we set $x = 0$, then we get $y = 0$, and if we set $x = 1$, then we get $y = 1$.

The graphs represented in Fig. 65 for $x > 0$ are located above the x-axis. This follows from the fact that if $x > 0$, then $x^n > 0$ for any natural n (see Sec. 22.1).

We also see that these graphs represent increasing (for nonnegative x) functions. This follows from the fact that if $0 \leqslant x_1 < x_2$, then $x_1^n < x_2^n$ for any natural n (see Sec. 22.1).

Finally, note the fact which we shall regard as obvious: if x, traversing positive values, tends to $+\infty$, then the corresponding value of the function y (obtained with the aid of one of Formulas (1) or (2)) also tends to $+\infty$, that is, if $x \to \infty$, then

$$x^2 \to +\infty; \quad x^3 \to +\infty; \quad x^4 \to +\infty; \quad \ldots \quad x^n \to +\infty.$$

On the interval (0, 1), that is, for the values of x for which

$$0 < x < 1,$$

the following inequalities are fulfilled:

$$x^2 > x^3 > x^4 \quad \ldots \tag{3}$$

This is obvious, since to get, for instance, x^3 from x^2, we have to multiply x^2 by x, where x is a positive number less than 1.

By virtue of Inequalities (3), the graph of the function $y = x^3$ on the interval (0, 1) is situated below the graph of the function $y = x^2$, and the graph of the function $y = x^4$ is found below the graph of the function $y = x^3$, and so on.

Further, on the interval (1, $+\infty$), that is, for the values of x for which $1 < x$ the inequalities

$$x^2 < x^3 < x^4 < \ldots, \tag{4}$$

are fulfilled since, for instance, x^3 is obtained from x^2 by multiplying the latter by the number $x > 1$. This shows that on the interval (1, $+\infty$) the parabola $y = x^3$ is situated above the parabola $y = x^2$, and the parabola $y = x^4$ is situated above the parabola $y = x^3$, and so forth.

SECTION TEST

QUESTIONS

1. What functions are called power functions?
2. What is the graph of the function $y = x^n$?
3. What is called a parabola of degree n?
4. Does the function $y = x^n$ increase for nonnegative x?

EXERCISES

1. Prove that the function $y = x^6$ increases for positive x.
2. Compile a table for the function $y = x^3$ with x changing every 0.5 on the interval [0, 5], and, using this table, draw an approximate graph of this function on the interval [0, 5].

Sec. 22.4. Graphs of Functions $y = x^n$

Let us now see what our parabolas look like for negative values of x, that is, for x's belonging to the interval $(-\infty, 0)$.

It is obvious that

$$(-x)^2 = x^2; \quad (-x)^4 = x^4; \quad (-x)^6 = x^6; \quad (-x)^8 = x^8,$$

and, in general, if $n = 2m$ is an even natural number, then

$$(-x)^{2m} = x^{2m}.$$

Indeed,

$$(-x)^{2m} = (-1 \cdot x)^{2m} = (-1)^{2m} x^{2m} = ((-1)^2)^m x^{2m}$$
$$= 1^m \cdot x^{2m} = 1 \cdot x^{2m} = x^{2m}.$$

This shows that the graph of the function $y = x^n$ for an even n is symmetric with respect to the y-axis. Such functions, as we know,

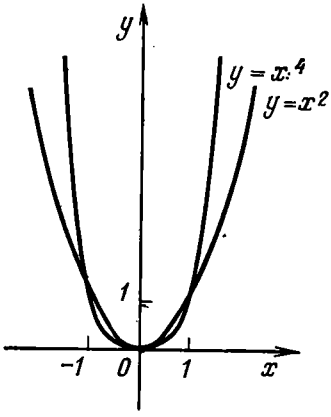


Fig. 66

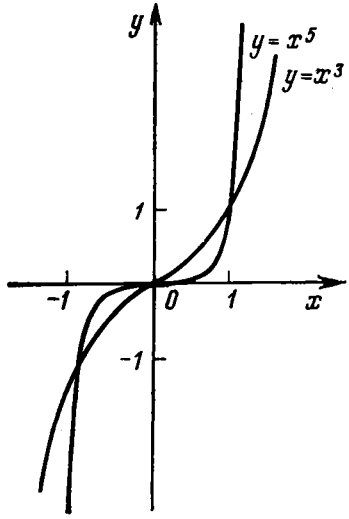


Fig. 67

are called *even*. Thus, the function $y = x^n$ for an even n is even. Figure 66 represents the graphs of the functions $y = x^2$ and $y = x^4$. Here x attains any real values.

And for power functions with odd powers there are fulfilled other equalities:

$$(-x)^3 = -x^3; \quad (-x)^5 = -x^5; \quad (-x)^7 = -x^7,$$

and, in general, if $n = 2m + 1$ is an odd natural number, then

$$(-x)^{2m+1} = -x^{2m+1}.$$

This shows that if n is an odd natural number and the point A (x, y) is a point of the parabola $y = x^n$, then the point A $(-x, -y)$ is also a point belonging to the same parabola which is symmetric to the given point with respect to the origin.

Consequently, for an odd n the parabola $y = x^n$ is a curve symmetric with respect to the origin. Using this property, it is not difficult to continue the drawn parabola for negative values of x.

Figure 67 shows the graphs of the functions $y = x^3$ and $y = x^5$. Here x also takes on any real values.

A function whose graph is symmetric with respect to the origin is termed *odd*. The aforegoing implies that the function $y = x^n$ for an odd n is odd.

Example 1. Prove that the function $y = x^3$ is continuous. To this end, let us assign any number x_0 and show that if

$$y = x^3, \quad y_0 = x_0^3$$

and $x - x_0 \to 0$, then $y - y_0 \to 0$.

Indeed,

$$\begin{aligned} y - y_0 = x^3 - x_0^3 &= [(x - x_0) + x_0]^3 - x_0^3 \\ &= (x - x_0)^3 + 3(x - x_0)^2 x_0 + 3(x - x_0) x_0^2 + x_0^3 - x_0^3 \\ &= (x - x_0)^3 + 3(x - x_0)^2 x_0 + 3(x - x_0) x_0^2. \end{aligned} \tag{1}$$

If $x - x_0$ tends to zero, that is, if

$$x - x_0 \to 0,$$

then each term in the right-hand side of Equalities (1) tends to zero, but then their sum also tends to zero, that is, if $x - x_0 \to 0$, then $y - y_0 \to 0$; this proves the continuity of the function $y = x^3$, and, consequently, the continuity of the graph of this function.

SECTION TEST

QUESTIONS

1. For what n are functions $y = x^n$ even and for what n odd?
2. For what n are parabolas $y = x^n$ symmetric with respect to the origin?
3. For what n are parabolas $y = x^n$ symmetric with respect to the y-axis?

EXERCISES

1. Prove that the function $y = x^4$ is continuous.
2. Compile a table for the function $y = x^3$ with x changing every 0.1 on the interval [0, 3], and, using this table, draw an approximate graph of the function $y = x^3$ on the interval [−3, 3].
3. Compile a table for the function $y = x^4$ with x changing every $\frac{1}{4}$ on the interval [0, 2], and, using this table, draw an approximate graph of the function $y = x^4$ on the interval [−2, 2].

Chapter 23

THE nth ROOT

Sec. 23.1. The Notion of the nth Root

Let a be an arbitrary real number, n a natural number, and $n \geqslant 2$. Raising a to the power n, we get a definite number which will be denoted by b. Thus,

$$a^n = b. \tag{1}$$

The number b, as we know, is called the nth power of the number a. And the number a is called the nth *root of* b. We also say that we have taken the nth root of b to get a.

In general, the nth root of the number b is defined as a number a (if it exists) such that its nth power is equal to b.

We already know that the second root of a number is called the square root of the number. The third root of a number is also called the cube root of the number.

The following examples

$$0^3 = 0; \quad 1^3 = 1; \quad 2^3 = 8; \quad 3^3 = 27; \quad \ldots;$$

$$(-1)^3 = -1; \quad (-2)^3 = -8; \quad (-3)^3 = -27;$$

show that the numbers

$$-3; \quad -2; \quad -1; \quad 0; \quad 1; \quad 2; \quad 3; \quad \ldots$$

can be regarded as respective cube roots of the numbers

$$-27; \quad -8; \quad -1; \quad 0; \quad 1; \quad 8; \quad 27; \quad \ldots.$$

And the examples

$$0^5 = 0; \quad 1^5 = 1; \quad 2^5 = 32; \quad 3^5 = 243; \quad \ldots;$$
$$(-1)^5 = -1; \quad (-2)^5 = -32; \quad (-3)^5 = -243; \quad \ldots$$

show that the numbers

$$-3; \quad -2; \quad -1; \quad 0; \quad 1; \quad 2; \quad 3; \quad \ldots$$

may also be considered as the fifth roots of the respective numbers

$$-243; \quad -32; \quad -1; \quad 0; \quad 1; \quad 32; \quad 243; \quad \ldots$$

For even n's somewhat different regularity takes place, for instance,

$$0^4 = 0; \quad 1^4 = 1; \quad 2^4 = 16; \quad 3^4 = 81; \quad \ldots;$$
$$(-1)^4 = 1; \quad (-2)^4 = 16; \quad (-3)^4 = 81; \quad \ldots,$$

whence it is seen that there are two numbers $+1$ and -1 which are the fourth roots of 1; there are two numbers $+2$ and -2 which are

the fourth roots of 16; there are also two numbers +3 and —3 which are the fourth roots of 81.

Further, 0 is the fourth root of 0.

We took (or extracted) the fourth roots of positive numbers. It is impossible to take the fourth root of negative numbers, that is, there is no fourth root of a negative number since the fourth power of any number different from zero is, obviously, a positive number.

The indicated facts will be put in order in the next sections, and we shall come to the following generalizations which are in accord with the above considered particular cases:

(1) If n is an odd number, then the nth root of any number exists and is a unique one. This is seen from the graph of the function $y = x^n$. An odd root of a positive number is a positive number, of a negative number a negative number, of zero is zero.

(2) If n is an even number, then there exist only two nth roots of any positive number; they differ only in sign; zero is the only nth root of zero. This is seen from the graph of the function $y = x^n$.

There is no even root of a negative number.

Remark. Later on we shall get acquainted with complex numbers and learn, in particular, that the even roots of negative numbers are complex numbers.

SECTION TEST

QUESTIONS

1. What is the square root of the number b?
2. What is the cube root of the number b?
3. What is the fifth root of the number b?
4. What is the nth root of the number b?
5. How many sixth roots of the number b are there?
6. How many ninth roots of the number b are there?

EXERCISES

1. Write all natural numbers whose cubes do not exceed 10,000.
2. Write all whole numbers whose fourth powers do not exceed 10,000.
3. Find out how many natural numbers are there whose sixth powers do not exceed 1,000,000.

Sec. 23.2. An Odd Root

Using the graphical method, let us show that for an odd natural n there exists the nth root of any real number b which is unique, that is, there exists a unique number a such that

$$a^n = b.$$

Here if $b > 0$, then $a > 0$; if $b < 0$, then $a < 0$; and if $b = 0$, then $a = 0$.

Let us draw in the Cartesian coordinate system Oxy the graph of the function

$$y = x^n. \tag{1}$$

Figure 68 represents the cubic parabola $y = x^3$; the general shape of the graph of the function $y = x^n$ for an odd natural n ($n \geqslant 3$) is the same as that of the cubic parabola. This is a continuous curve passing through the origin and symmetric with respect to the origin.

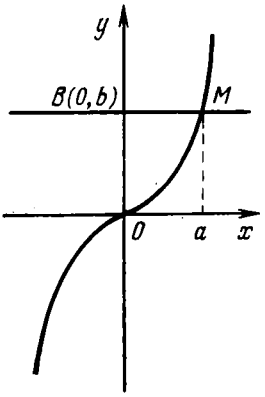


Fig. 68

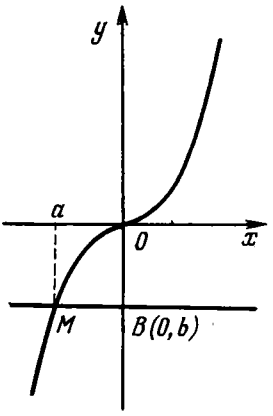


Fig. 69

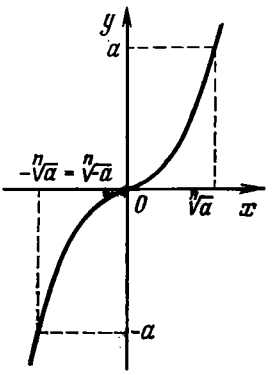


Fig. 70

Let us be given an arbitrary number b. Through the point B $(0, b)$ draw a straight line parallel to the x-axis. This line will intersect the nth-degree parabola $y = x^n$ at one point M. This point has the ordinate $y = b$ and abscissa $x = a$.

Figure 68 illustrates the case when $b > 0$; then a is positive.

Figure 69 represents the case when $b < 0$; then a is negative. It is also seen from the graph that if $b = 0$, then $a = 0$. In all these cases the equality

$$a^n = b \tag{2}$$

holds true. Consequently, it has been shown that for any number b there exists only one number a satisfying Condition (2). This number a is just called the nth root of the number b.

The symbol for a root is the sign $\sqrt{\ }$, called the radical sign.

For an odd n the nth root of any number b is denoted as follows:

$$a = \sqrt[n]{b}. \tag{3}$$

For instance,

$$\sqrt[3]{8} = 2; \quad \sqrt[3]{-8} = -2; \quad \sqrt[3]{27} = 3; \quad \sqrt[3]{-27} = -3;$$
$$\sqrt[5]{100,000} = 10; \quad \sqrt[5]{-100,000} = -10.$$

The following properties of odd nth roots hold true: for any real numbers a, b, and c ($c \neq 0$):

$$(1)\ \sqrt[n]{-a} = -\sqrt[n]{a};\quad (2)\ (\sqrt[n]{a})^n = a;\quad (3)\ \sqrt[n]{a^n} = a;$$

$$(4)\ \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b};\quad (5)\ \sqrt[n]{\frac{a}{c}} = \frac{\sqrt[n]{a}}{\sqrt[n]{c}}.$$

Equality (1) follows from the fact that the graph of the function $y = x^n$ is symmetric with respect to the origin for an odd n (Fig. 70). For instance,

$$\sqrt[3]{-27} = -\sqrt[3]{27} = -3;\quad \sqrt[3]{-8} = -\sqrt[3]{8} = -2;$$

$$\sqrt[5]{-1} = -\sqrt[5]{1} = -1;\quad \sqrt[5]{-100,000} = -\sqrt[5]{100,000} = -10.$$

Equality (2) is true by the definition of an odd root of a real number. For instance,

$$(\sqrt[5]{3})^5 = 3;\quad (\sqrt[5]{-3})^5 = -3;\quad (\sqrt[3]{-1})^3 = -1;\quad (\sqrt[3]{-27})^3 = -27.$$

Equality (3) is true since it is obvious that a is the nth root of the number a^n: raising a to nth power, we just obtain a^n. For instance,

$$\sqrt[3]{2^3} = 2;\quad \sqrt[5]{3^5} = 3;\quad \sqrt[3]{(-2)^3} = -2;\quad \sqrt[5]{(-3)^5} = -3.$$

As an implication of Property (3), we get the following statement: if for real numbers a and b and for an odd natural n the equality

$$a^n = b^n$$

is true, then the equality

$$a = b$$

is also true.

Taking advantage of this statement, let us prove Property (4). By Property (2),

$$(\sqrt[n]{ab})^n = ab.$$

By the property of powers and Property (2),

$$(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n (\sqrt[n]{b})^n = ab.$$

The right-hand members of these equalities are equal, hence their left-hand members are also equal to each other, that is,

$$(\sqrt[n]{ab})^n = (\sqrt[n]{a}\sqrt[n]{b})^n,$$

hence, we get

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$$

Property (5) is proved in a similar way.

Examples.

$$\sqrt[3]{24}=\sqrt[3]{8\cdot 3}=\sqrt[3]{8}\sqrt[3]{3}=2\sqrt[3]{3};$$

$$\sqrt[3]{-54}=-\sqrt[3]{27\cdot 2}=-\sqrt[3]{27}\sqrt[3]{2}=-3\sqrt[3]{2};$$

$$\sqrt[3]{\frac{5}{8}}=\frac{\sqrt[3]{5}}{\sqrt[3]{8}}=\frac{\sqrt[3]{5}}{2};\quad \sqrt[3]{-\frac{2}{9}}=-\sqrt[3]{\frac{6}{27}}=-\frac{\sqrt[3]{6}}{\sqrt[3]{27}}=-\frac{\sqrt[3]{6}}{3}.$$

Note that if b is a nonnegative number, then the nth root of b, that is,

$$a=\sqrt[n]{b},$$

is also called the *arithmetic value of the nth root of b* or the *nth arithmetic root of b*.

As it was shown above, if $b\geqslant 0$, then $a\geqslant 0$.

SECTION TEST

QUESTIONS

1. For what real numbers does an odd nth root exist?
2. How many odd nth roots of a given number are there?
3. What are the properties of odd nth roots?

EXERCISES

1. Simplify the following numerical expressions:

(a) $\sqrt[3]{32}$; (b) $\sqrt[5]{800}$; (c) $30\sqrt[3]{\frac{1}{12}}+\frac{7}{2}\sqrt[3]{\frac{2}{3}}+5\sqrt[3]{144}$;

(d) $\sqrt[3]{48}-\sqrt[3]{75}+\sqrt[3]{32}-\sqrt[3]{40}$.

Sec. 23.3. An Even Root

Applying the graphical method, let us show that for an even natural n there are two and only two nth roots of any positive number. They differ only in sign. The nth root of 0 is unique and equals 0. As far as negative numbers are concerned, even roots of them do not exist.

When raised to an even power, any number other than zero yields a positive number. When raised to the nth power, the number zero gives zero. This just proves the last two statements.

Let us now show that the first statement is also true. To this end, consider the function

$$y=x^n. \tag{1}$$

For an even natural n its graph resembles a second-degree parabola (Fig. 71). This is a continuous curve passing through the origin and symmetric with respect to the y-axis.

Let us be given an arbitrary positive number b. Through the point $B\ (0,\ b)$ we draw a straight line parallel to the x-axis. This line intersects the graph of the function $y = x^n$ at two points: M and N. Let the abscissa of the point M be the number a, then the abscissa of the point N will be represented by $(-a)$. It is obvious that the nth powers of a and $(-a)$ are equal to b:

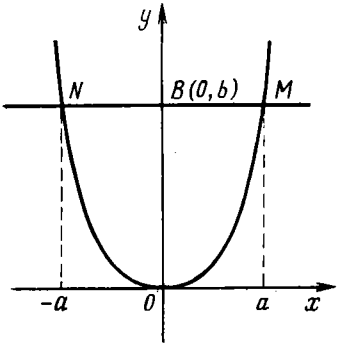


Fig. 71

$$a^n = (-a)^n = b.$$

Thus, for an even n, for every positive number b there exist exactly two nth roots of b. One of them is positive, we shall assume that this is a, the other being equal to $(-a)$. For the number a the following notation is introduced:

$$a = \sqrt[n]{b}.$$

This notation is extended to $b = 0$ (then $a = 0$) and is called the *arithmetic value of the* nth *root of* b or, simply, the nth *arithmetic root of* b.

And the other, negative, nth root of b $(b > 0)$ is equal to

$$-\sqrt[n]{b}.$$

Thus, to write a negative even nth root of a positive number b, it is necessary to put the minus sign in front of the arithmetic value of the nth root of b.

Hence, if b is a positive number, then there are two even nth roots of it:

$$\sqrt[n]{b} \text{ and } -\sqrt[n]{b}.$$

For example,

$$\sqrt[4]{32} = 2; \quad \sqrt[6]{1{,}000{,}000} = 10; \quad \sqrt{0} = 0$$

are arithmetic values of the respective roots.

The expressions

$$\sqrt[4]{17}; \quad \sqrt[6]{19}; \quad \sqrt[8]{25}; \quad \sqrt[4]{0.(12)}$$

also represent arithmetic values of respective roots. If necessary, they would be computed approximately.

The expressions

$$\sqrt[4]{-16};\quad \sqrt[4]{-81};\quad \sqrt[6]{-1{,}000{,}000};\quad \sqrt[4]{-13.2};\quad \sqrt[8]{-0.1},$$

as real numbers, have no sense since there is no even root of a negative number.

The expressions

$$-\sqrt[4]{16}=-2;\quad -\sqrt[4]{17};\quad -\sqrt[4]{81}=-3$$

are negative fourth roots of the numbers 16, 17, 81, respectively.

The following properties of even *n*th roots hold true:

(1) for any nonnegative number a

$$(\sqrt[n]{a})^n=a;$$

(2) for any real number a

$$\sqrt[n]{a^n}=|a|;$$

(3) for any nonnegative numbers a and b

$$\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b};$$

(4) for any nonnegative number a and positive b

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

The number $\sqrt[n]{a}$ is the *n*th root of a. Therefore, raising it to the *n*th power, we get a. This is affirmed by Property (1). Since in this case n is even, we have to assume that $a\geqslant 0$.

Examples.

$$(\sqrt[4]{2})^4=2;\quad (\sqrt{3})^2=3;\quad (\sqrt[6]{7})^6=7;\quad (\sqrt[4]{1})^4=1.$$

In the equality of Property (2) the number a may be arbitrary. On raising it to an even power n, we get $a^n\geqslant 0$, and the expression $\sqrt[n]{a^n}$ has sense.

For $a=0$ the equality of Property (2) is obvious. And if $a\neq 0$, we reason as follows: the number a, is, obviously, the *n*th root of a^n and, consequently, the other root is $-a$. Since the left-hand member of the equality of Property (2) is a positive *n*th root of a^n, it is equal to one of the numbers a and $-a$ which is positive, that is, it is equal to the number $|a|$.

Examples.

$$\sqrt[4]{2^4}=2;\quad \sqrt[4]{(-2)^4}=2;\quad \sqrt{(-3)^2}=3.$$

As an implication of Property (2), we have the following statement: for nonnegative numbers a and b and an even natural n from the equality

$$a^n=b^n$$

there follows the equality

$$a = b.$$

Using this statement, let us prove Property (3). Since a and b are nonnegative numbers, the following expressions have sense:

$$\sqrt{a};\quad \sqrt[n]{b};\quad \text{and}\quad \sqrt[n]{ab}.$$

By Property (1),

$$(\sqrt[n]{ab})^n = ab,\quad (\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab.$$

The right-hand members of these equalities are equal, hence,

$$(\sqrt[n]{ab})^n = (\sqrt[n]{a}\sqrt[n]{b})^n.$$

Since for even n the numbers in the parentheses of the last equalit, are nonnegative, taking advantage of the aforegoing implicationy we get the equality of Property (3).

The equality of Property (4) is proved in a similar way.

Examples.

$$\sqrt[4]{48} = \sqrt[4]{16\cdot 3} = \sqrt[4]{16}\sqrt[4]{3} = 2\sqrt[4]{3};\quad \sqrt{3}\sqrt{27} = \sqrt{81} = 9;$$

$$\sqrt{\frac{25}{12}} = \frac{\sqrt{25}}{\sqrt{12}} = \frac{5}{\sqrt{4}\sqrt{3}} = \frac{5}{2\sqrt{3}}.$$

SECTION TEST

QUESTIONS

1. For what real numbers is there an even nth root?
2. How many even nth roots of a given number are there?
3. What are the properties of even nth roots?

EXERCISES

1. Simplify the following numerical expressions:

(a) $\sqrt[4]{80}$; (b) $\sqrt[4]{405}$; (c) $\sqrt[4]{81(4-\sqrt{17})^4}$; (d) $\sqrt[4]{0.0001} - \sqrt[6]{0.000064}$.

Sec. 23.4. The nth Arithmetic Root

In order to avoid misunderstandings, let us recall that the symbol $\sqrt[n]{b}$ means:

(1) the nth root of b if it is unique, or

(2) a positive nth root of b if the number b has two nth roots.

Case (1) occurs if n is an odd number for any b, and if n is an even number, then for $b = 0$.

For example,

$$\sqrt[3]{27} = 3;\quad \sqrt[3]{-27} = -3;\quad \sqrt[5]{-1} = -1;\quad \sqrt[7]{0} = 0;\quad \sqrt[8]{0} = 0.$$

Case (2) happens if n is an even number only for positive b.
For example,

$$\sqrt[4]{16}=2; \quad \sqrt{25}=5; \quad \sqrt[4]{81}=3.$$

The expression $\sqrt[n]{b}$ for an even n and $b<0$ is regarded as having no sense. Later on, when complex numbers will be introduced, such an expression will acquire sense not as a real, but as a complex number.

For example, the following expressions

$$\sqrt{-2}; \quad \sqrt[4]{-16}; \quad \sqrt[4]{-3}; \quad \sqrt[6]{-17}, \quad \sqrt[6]{-1,000,000}$$

are regarded as having no sense since they do not define real numbers.

The expression $\sqrt[n]{b}$ for nonnegative b is also called an *arithmetic value of the nth root of b* or the *nth arithmetic root of b*.

The aforegoing implies that the nth arithmetic root has sense both for an even and odd n.

We may also give such a definition: the nth arithmetic root of a number b is defined as a nonnegative nth root of this number.

It is not stated in this definition that b is a nonnegative number. But only nonnegative numbers b have nonnegative roots.

The numbers

$$\sqrt{4}=2; \quad \sqrt[3]{27}=3; \quad \sqrt[5]{1}=1; \quad \sqrt[3]{17}; \quad \sqrt[5]{21}; \quad \sqrt[4]{18}$$

may serve as examples of arithmetic values of the respective roots.

In Secs. 23.2 and 23.3 we studied the properties of even and odd roots. They turned out to be different. But if we consider nth arithmetic roots, then we shall see that they have a number of properties in common, irrespective of whether n is even or odd. It is easy to see that the results obtained in the two previous sections imply the following properties of nth arithmetic roots.

Let a and b be any nonnegative numbers, c any positive number, and n a natural number exceeding 1 ($n \geqslant 2$). Then the following equalities hold true:

(1) $(\sqrt[n]{a})^n=a$; (2) $\sqrt[n]{a^n}=a$; (3) $\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$; (4) $\sqrt[n]{\frac{a}{c}}=\frac{\sqrt[n]{a}}{\sqrt[n]{c}}$.

Let us also note that the results of the above mentioned sections imply the following statement: if a and b are nonnegative numbers and n is a natural number, then from the equality

$$a^n=b^n$$

there follows the equality

$$a=b.$$

Making use of this statement, we may also prove a number of properties of nth arithmetic roots.

Let a be any nonnegative number, m, n, and k natural numbers ($n \geqslant 2$, $k \geqslant 2$), then the following equalities hold true:

$$(5)\ (\sqrt[n]{a})^m = \sqrt[n]{a^m};\quad (6)\ \sqrt[mn]{a^m} = \sqrt[n]{a};\quad (7)\ \sqrt[k]{\sqrt[n]{a}} = \sqrt[kn]{a}.$$

Let us prove, for instance, Equality (5). It is easy to see that

$$[(\sqrt[n]{a})^m]^n = [(\sqrt[n]{a})^n]^m = a^m;\quad (\sqrt[n]{a^m})^n = a^m.$$

From the equality of the right-hand members there follows the equality of the left-hand members, and then, on the basis of the above statement, Equality (5) has been proved.

Equalities (6) and (7) are proved in a similar way.

Examples.

$$\sqrt{4^3} = (\sqrt{4})^3 = 2^3;\quad \sqrt[3]{\frac{1}{27^4}} = \left(\sqrt[3]{\frac{1}{27}}\right)^4 = \frac{1}{3^4} = \frac{1}{81};$$

$$\sqrt[3]{\sqrt{12}} = \sqrt[6]{12};\quad \sqrt{\sqrt{16}} = \sqrt[4]{16} = 2;$$

$$\sqrt[6]{2^3} = \sqrt{2};\quad \sqrt[4]{64} = \sqrt[4]{2^6} = \sqrt{2^3} = 2\sqrt{2}.$$

Problem. Solve the equation

$$x^n = A,\quad ;$$

where n is a given natural number and A is a given real number, that is, find all such numbers x whose nth powers are equal to A. In other words, we have to find all nth roots of the number A.

If $A < 0$ and n is even, then, obviously, this equation has no solution.

Excluding this case, we shall assume that the solution x is substituted into this equation. Let us extract the nth root of it, that is, let us take the nth root of both of its sides.

We get

$$\sqrt[n]{x^n} = \sqrt[n]{A}.$$

For an odd n we get

$$x = \sqrt[n]{A}.$$

And for an even n we have

$$|x| = \sqrt[n]{A}.$$

Hence, for an odd n and any real A the equation has the only root

$$x_0 = \sqrt[n]{A};$$

for an even n and $A > 0$ the equation has two roots:

$$x_1 = \sqrt[n]{A} \quad \text{and} \quad x_2 = -\sqrt[n]{A}.$$

Finally, for an even n and $A = 0$ the equation has the only root $x_0 = 0$.

Remark. Henceforward, when considering the nth arithmetic root, we shall add in words that this is an arithmetic root or write after the root symbol $\sqrt[n]{\ }$ that $a \geqslant 0$. In other words, the notation $\sqrt[n]{a}$ $(a \geqslant 0)$ will mean that we consider the nth arithmetic root of the number a.

SECTION TEST

QUESTIONS

1. What is the nth arithmetic root of the number a?
2. How many nth arithmetic roots of a given number are there?
3. What are the properties of nth arithmetic roots?

EXERCISES

1. Simplify the following numerical expressions:

(a) $\sqrt{2\sqrt[4]{4\sqrt{4}}}$; (b) $\sqrt{2\sqrt[3]{2}} : \sqrt[3]{2\sqrt{2}}$; (c) $\sqrt[3]{32\sqrt[4]{2}} \cdot \sqrt{2\sqrt[4]{4\sqrt[3]{4}}}$.

2. Write the following three numbers in the form of roots of the same index:

(a) $\sqrt[3]{3}$, $\sqrt{2}$ and $\sqrt[6]{5}$; (b) $\sqrt{5}$, $\sqrt[4]{15}$ and $\sqrt[8]{50}$.

Sec. 23.5. The nth Root of a Natural Number

Let n denote a natural number $(n \geqslant 2)$. It is obvious that the nth power of a natural number b is a natural number. But not any natural number is the nth power $(n \geqslant 2)$ of a certain natural number.

For instance, among the natural numbers not exceeding 100 only 4 (that is, 4%) are the cubes of natural numbers, namely: 1^3, 2^3, 3^3, 4^3. Among the natural numbers not exceeding 1000, only 10, that is, 1%, are the cubes of natural numbers, namely: 1^3, 2^3, 3^3, . . ., 10^3.

We see that nth powers of natural numbers seldom occur among large natural numbers.

Let us note the following fact: if a natural number is not an nth $(n \geqslant 2)$ power of some natural number, then it is an nth power of an irrational number, that is, it cannot be the nth power of a rational number.

Thus, the following roots:

$$\sqrt{2}; \quad \sqrt[3]{5}; \quad \sqrt[4]{3}; \quad \sqrt[5]{7}; \quad \sqrt[6]{19}$$

are irrational numbers.

Of course, this statement should be proved for any $n \geqslant 2$. It is done in the same way as it was proved in Chapter 11 for $n = 2$. But we are not going to dwell on this proof.

If a given natural number is not the nth power of a natural number, then we are used to say that the nth root of this number is inextractable.

Let us illustrate by an example how it is possible to take approximately the nth root of a given number. Let us compute approximately the number $\sqrt[3]{17}$ to within the second decimal place.

We know that this number is positive; it has a certain decimal expansion:

$$\sqrt[3]{17} = \alpha_0 \cdot \alpha_1\alpha_2\alpha_3 \ldots .$$

We have to find the numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$. Consider the numbers

$$0^3;\ 1^3;\quad 2^3;\quad 3^3;\quad \ldots \tag{1}$$

to find two successive numbers between which the number 17 is found. Obviously

$$8 = 2^3 < 17 < 3^3 = 27.$$

Consequently,

$$2 < \sqrt[3]{17} < 3, \qquad \alpha_0 = 2.$$

Now, consider the numbers

$$2^3;\quad 2.1^3;\quad 2.2^3;\quad 2.4^3;\quad \ldots;\quad 2.9^3;\quad 3^3$$

to find two successive numbers between which the number 17 is found.

We have

$$15.625 = 2.5^3 < 17 < 2.6^3 = 17.676,$$

whence

$$2.5 < \sqrt[3]{17} < 2.6,$$

and, consequently,

$$\alpha_1 = 5.$$

Now, for the same purpose, consider the numbers

$$2.5^3;\quad 2.51^3;\quad 2.52^3;\quad \ldots;\quad 2.59^3;\quad 2.6^3.$$

It turns out that

$$16.974\ldots = 2.57^3 < 17 < 2.58^3 = 17.173\ldots,$$

consequently,

$$\alpha_2 = 7.$$

Thus,

$$\sqrt[3]{17} = 2.57\ldots .$$

We see that the computation method is primitive, but awkward. Recently, hand-held calculators are widely used. With the aid of

such a calculator the cube root of 17 can be taken in no time. The accuracy of the obtained approximate value depends on the rated accuracy of the calculator used.

Approximate values of the square and cube roots of numbers can be obtained from the tables of squares and cubes of numbers.

Let us also note that we can compute any roots of numbers with the aid of a slide rule and tables of logarithms. But this will be discussed later on.

SECTION TEST

QUESTIONS

1. Can the nth root ($n \geqslant 2$) of a prime number be a rational number?
2. Can the nth root ($n \geqslant 2$) of a natural number be a rational number?
3. What does it mean to compute $\sqrt[n]{N}$ to within the third decimal place, where N is a prime number?

EXERCISES

1. Compute the following numbers to within the third decimal place:
(a) $\sqrt[3]{3}$; (b) $\sqrt[3]{5}$; (c) $\sqrt[3]{7}$
and check the obtained results against table of cube roots.
2. Compute the following numbers to within the first decimal place:
(a) $\sqrt[4]{3}$; (b) $\sqrt[5]{7}$.

Sec. 23.6. The Function $y = \sqrt[n]{x}$

Let n be a natural number, and let $n \geqslant 2$. Consider the function

$$y = x^n$$

for nonnegative values of x only. The graph of the function

$$y = x^n \qquad (x \geqslant 0) \tag{1}$$

in the Cartesian coordinate system Oxy is represented by a branch of a parabola of degree n shown in Fig. 72. It is easy to see that if we assign an arbitrary nonnegative value of y, mark on the y-axis the point B $(0, y)$, and erect at this point a perpendicular to the y-axis to intersect Parabola (1), then the point of intersection will be found at A (x, y) whose abscissa is $x = \sqrt[n]{y}$.

Thus, by Formula (1), every nonnegative number y may be associated with a unique nonnegative number x such that $x = \sqrt[n]{y}$. In other words, if y is regarded as an independent variable and x as a dependent variable, then on the set $y \geqslant 0$ we have defined the function

$$x = \sqrt[n]{y} \qquad (y \geqslant 0). \tag{2}$$

This function brings every nonnegative number y to the correspondence with a nonnegative number x the nth power of which is y such that $y = x^n$.

It is clear that the parabola represented in Fig. 72 is not only the graph of Function (1) in the coordinate system Oxy, but also the graph of Function (2) in the coordinate system Oyx, that is, if the y-axis is taken for the axis of the independent variable, and the x-axis for the axis of the dependent variable.

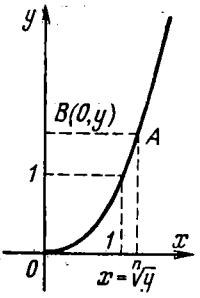


Fig. 72

Thus, the graph of the function $x = \sqrt[n]{y}$ $(y \geqslant 0)$ in the coordinate system Oyx is a parabola which is the graph of the function $y = x^n$ in the coordinate system Oxy.

In our considerations, the letters x and y may be interchanged, and we may consider the function

$$y = \sqrt[n]{x} \qquad (x \geqslant 0). \tag{3}$$

To obtain its graph in the coordinate system Oxy, it is necessary to construct the graph of the function $x = y^n$ $(y \geqslant 0)$ in the coordinate system Oyx (Fig. 73). This graph is just the graph of Function (3) in the coordinate system Oxy.

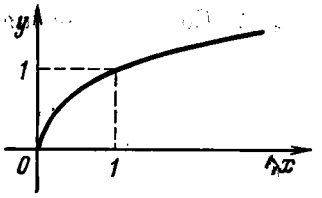


Fig. 73

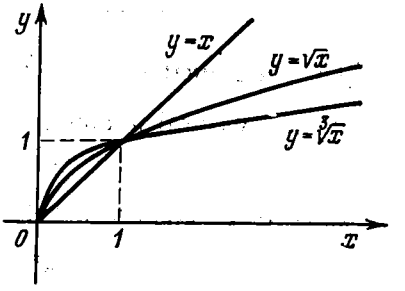


Fig. 74

We see that the graph of Function (3) is a continuous curve emanating from the origin and passing through the point (1, 1). The ordinate y of its moving point (x, y) increases with an increase in the abscissa x so that $y \to +\infty$ as $x \to +\infty$.

From the graph it is also seen that

$$\sqrt[n]{x} < 1 \quad \text{for} \quad 0 \leqslant x < 1$$

and

$$\sqrt[n]{x} > 1 \quad \text{for} \quad x > 1.$$

Figure 74 represents in the same Cartesian coordinate system Oxy the graphs of the following functions:

$$y = x; \quad y = \sqrt{x}; \quad y = \sqrt[3]{x} \quad (x \geqslant 0).$$

Here, we see an interesting regularity:
if $0 < x < 1$, then $x < \sqrt{x} < \sqrt[3]{x} < \sqrt[4]{x} < \ldots$; and if $x > 1$, then $x > \sqrt{x} > \sqrt[3]{x} > \sqrt[4]{x} > \ldots$. For instance, if $0 < x < 1$, then $(\sqrt{x})^6 = x^3 < x^2 = (\sqrt[3]{x})^6$, whence $\sqrt{x} < \sqrt[3]{x}$.

SECTION TEST

QUESTIONS

1. How is it possible to draw the graph of the function $y = \sqrt[n]{x}$ $(x \geqslant 0)$?
2. Is the function $y = \sqrt[n]{x}$ $(x \geqslant 0)$ continuous?
3. Is the function $y = \sqrt[n]{x}$ $(x \geqslant 0)$ decreasing?

EXERCISES

1. Prove that the function $y = \sqrt[n]{x}$ $(x \geqslant 0)$ increases.

Chapter 24

THE FUNCTION $y = \frac{1}{x}$

Sec. 24.1. Properties of the Function $y = \frac{1}{x}$

The function

$$y = \frac{1}{x}$$

is defined for any numerical values of x except for $x = 0$.
First we shall consider this function only for positive x's:

$$y = \frac{1}{x} \qquad (x > 0).$$

Note the following of its properties:
(1) If $x > 0$, then $y > 0$.
(2) For positive values of x the function $y = \frac{1}{x}$ decreases.

In other words, for positive x's with an increase in x, y decreases, that is, if $0 < x_1 < x_2$, then $y_1 > y_2$, where $y_1 = \frac{1}{x_1}$, $y_2 = \frac{1}{x_2}$. This statement follows from Property (5) of numerical inequalities (see Sec. 3.6).

(3) If a positive x tends to zero, then $y=\frac{1}{x}$ tends to $+\infty$, and if x tends to $+\infty$, then $y=\frac{1}{x}$ tends to zero, that is,

$$y=\frac{1}{x}\to+\infty \quad \text{as} \quad x\to 0 \quad (x>0);$$

$$y=\frac{1}{x}\to 0 \qquad \text{as} \quad x\to+\infty.$$

We shall illustrate these properties by examples.

If a positive x tends to zero running through the values

$$x=1;\ \frac{1}{2};\ \frac{1}{3};\ \frac{1}{4};\ \ldots,$$

then the function $y=\frac{1}{x}$ traverses accordingly the values

$$y=1;\ 2;\ 3;\ 4;\ \ldots,$$

that is, $y\to+\infty$.

And if x tends to $+\infty$, running through the values

$$x=1;\ 2;\ 3;\ 4;\ \ldots,$$

then, accordingly, y tends to zero, traversing the values

$$y=1;\ \frac{1}{2};\ \frac{1}{3};\ \frac{1}{4};\ \ldots\ .$$

(4) If a positive x tends to a given positive number x_0, then the function $y=\frac{1}{x}$ tends to the number $\frac{1}{x_0}$, that is,

$$\frac{1}{x}\to\frac{1}{x_0} \quad \text{as} \quad x\to x_0$$

or

$$(y-y_0)\to 0 \quad \text{as} \quad (x-x_0)\to 0.$$

Indeed,

$$y-y_0=\frac{1}{x}-\frac{1}{x_0}=\frac{x_0-x}{x_0x}\to 0 \text{ as } x-x_0\to 0.$$

In fact, if $x-x_0$ tends to zero, then the numerator of the fraction $\frac{x_0-x}{xx_0}$ tends to zero, while the denominator tends to the positive number x_0^2, but then the fraction tends to zero. This property is called the property of continuity of the function $y=\frac{1}{x}$ for positive x's.

5) The function $y=\frac{1}{x}$ is odd. Indeed, with the sign of x reversed, the sign of the corresponding value of the function $y=\frac{1}{x}$ is also reversed.

(6) For positive x's the function $y=\frac{1}{x}$ is inversely proportional to the argument x or, in other words, if $y_1=\frac{1}{x_1}$ and $y_2=\frac{1}{x_2}$, then $x_1 : x_2 = y_2 : y_1$. Indeed,

$$y_2 : y_1 = \frac{1}{x_2} : \frac{1}{x_1} = x_1 : x_2.$$

SECTION TEST

QUESTIONS

1. For what x's does the function $y=\frac{1}{x}$ decrease?
2. Is the function $y=\frac{1}{x}$ continuous?
3. Is the function $y=\frac{1}{x}$ odd?
4. Is the function $y=\frac{1}{x}$ decreasing for positive x's?
5. Is the function $y=\frac{1}{x}$ continuous for negative x's?

EXERCISES

1. Prove that the function $y=\frac{1}{x}$ decreases for negative x's.
2. Find out to what does the function $y=\frac{1}{x}$ tend if negative x's tend to zero.
3. Prove that the function $y=\frac{1}{x}$ is continuous for negative x's.

Sec. 24.2. The Graph of the Function $y=\frac{1}{x}$

Let us first draw the graph of the function $y=\frac{1}{x}$ for positive x's. To this end, we must compute for every positive number x the corresponding value of y by the formula $y=\frac{1}{x}$ and plot the points (x, y) thus obtained on the plane with the Cartesian coordinate system specified.

The totality of all these points forms the graph of the function $y=\frac{1}{x}$ for positive x's.

But it is impossible to complete this process since there are infinitely many such points. But still we can draw approximately the graph of our function.

Let us take just a little more separate positive values of x and compute the corresponding values of y by the formula $y = \frac{1}{x}$. Given below is such a table for some values of x:

x	y	x	y	x	y
0.2	5	0.6	10/6	1	1
0.3	10/3	0.7	10/7	2	1/2
0.4	10/4	0.8	10/8	3	1/3
0.5	10/5	0.9	10/9	4	1/4

We plot the points contained in the table in a plane with the given Cartesian coordinate system *Oxy* (Fig. 75) and connect these points with a smooth continuous curve such that the ordinate y of its

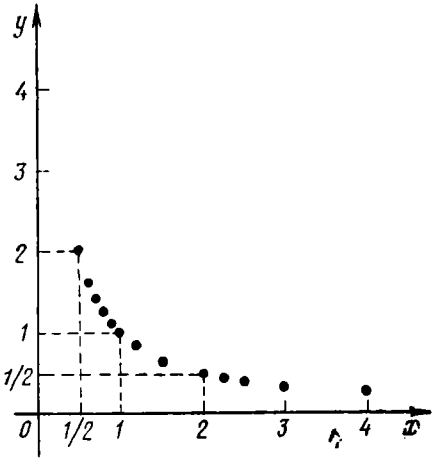


Fig. 75

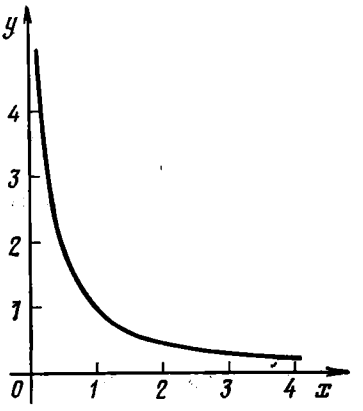


Fig. 76

moving point decreases with an increase in its abscissa. The continuous curve thus obtained may be regarded as an approximate graph of the function $y = \frac{1}{x}$ for positive x's.

Note that the graph represented in Fig. 76 reflects Properties (1), (2), (3), and (4) of the function $y = \frac{1}{x}$ formulated in the preceding section.

The graph of the function $y = \frac{1}{x}$ for positive x's is entirely situated above the x-axis. This just reflects Property (1).

With an increase in the abscissa of the point x moving in the curve, its ordinate decreases. This illustrates Property (2). And if $x \to \infty$, then $y \to 0$. If $x \to 0$, then $y \to +\infty$. This Property (Property (3)) is also obvious from Fig. 76.

Finally, by virtue of Property (4), our graph must be a continuous line, therefore we just connected the plotted points with a continuous curve. If A is a point of the continuous graph having the abscissa x_0 and ordinate y_0 and B is another point of the graph with abscissa x and ordinate y, then as x tends to x_0, the point B, moving along the graph, tends, that is, approaches arbitrarily close to the point A. Consequently, the ordinate y of the point B tends to the ordinate y_0 of the point A. But this is exactly Property (4) of the function $y = \frac{1}{x}$. By virtue of this property, $\frac{1}{x} \to \frac{1}{x_0}$ as $x \to x_0$.

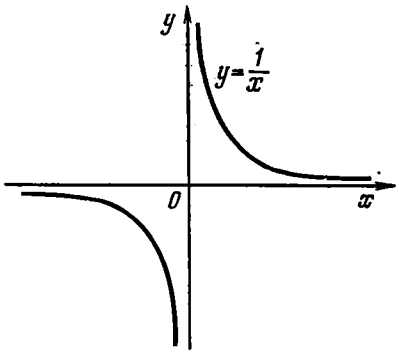


Fig. 77

As we know, the graph of an odd function is symmetric with respect to the origin. Therefore, to construct the graph of the function $y = \frac{1}{x}$ for negative x's, it is necessary to draw a curve symmetric to the already constructed curve with respect to the origin.

The graph of the function $y = \frac{1}{x}$ for all x's different from zero has the form represented in Fig. 77.

The curve representing the graph of the function $y = \frac{1}{x}$ is called the *hyperbola*. We also say that hyperbola is the graph of the function $y = \frac{1}{x}$ and has the equation $y = \frac{1}{x}$. We see that the hyperbola $y = \frac{1}{x}$ consists of two branches: one for negative values of x, the other for positive. Let us now find out the form of the graph of the function

$$y = \frac{k}{x},$$

where k is a given number different from zero.

If $k > 0$, then the graph of the function $y = \frac{k}{x}$ is a curve resembling the graph of the hyperbola $y = \frac{1}{x}$. Figure 78 represents the graphs

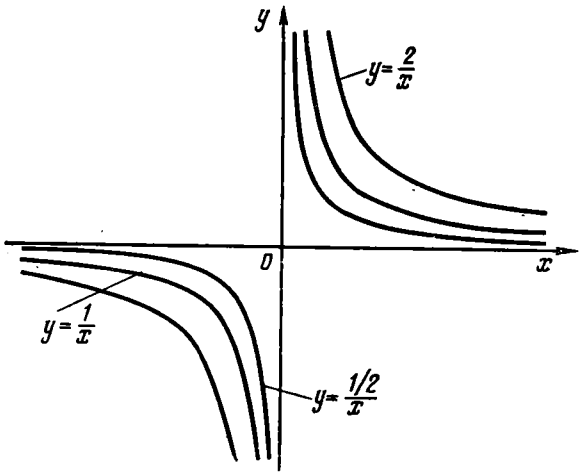


Fig. 78

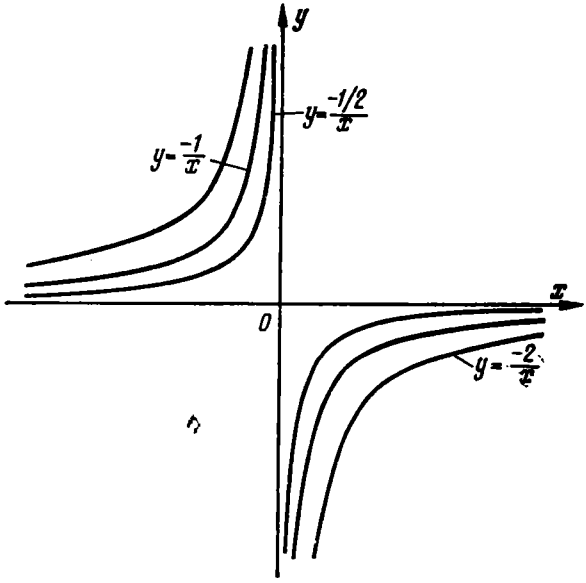


Fig. 79

of the functions: $y=\frac{1}{x}$; $y=\frac{1/2}{x}$; $y=\frac{2}{x}$. And if $k<0$, then the graph of the function

$$y=\frac{k}{x}$$

is symmetric with respect to the x-axis to the graph of the function

$$y=\frac{k_1}{x},$$

where $k_1=-k>0$. Figure 79 represents the graphs of the functions

$$y=\frac{-1}{x};\quad y=\frac{-1/2}{x};\quad y=\frac{-2}{x}.$$

The graphs of the functions

$$y=\frac{k}{x}\qquad (k\neq 0)$$

are also called hyperbolas.

SECTION TEST

QUESTIONS

1. What is the graph of the function $y=\frac{1}{x}$?
2. What eguation does the hyperbola have?
3. How many branches does a hyperbola consist of?

EXERCISES

1. Represent the graphs of the indicated functions in the same coordinate system:

$$y=\frac{3}{x};\ y=\frac{-3}{x};\ y=\frac{1/3}{x};\ y=\frac{-1/3}{x}.$$

Sec. 24.3. The Function $y=\frac{1}{x^n}$

The function

$$y=\frac{1}{x^n},$$

where n is a given natural number, is defined for any numerical values of x except for $x=0$.

First, we list the properties of this function only for positive x's:

(1) If $x>0$, then $y>0$.

(2) For positive values of x the function $y=\frac{1}{x^n}$ decreases.

(3) If a positive x tends to zero, then $y=\frac{1}{x^n}$ tends to $+\infty$, and if x tends to $+\infty$, then $y=\frac{1}{x^n}$ tends to zero, that is,

$$y=\frac{1}{x^n}\to+\infty \text{ as } x\to 0\ (x>0),$$

$$y=\frac{1}{x^n}\to 0 \quad \text{as } x\to+\infty.$$

(4) For positive x's the function $y=\frac{1}{x^n}$ is continuous.

These properties are proved in the same manner as Properties (1)-(4) in Sec. 24.1.

To construct the graph of the function $y=\frac{1}{x^n}$ for positive x's, we just take a sufficient number of separate values of x and compute the corresponding values of y by the formula $y=\frac{1}{x^n}$. We then plot these points on the coordinate plane and connect them with a smooth continuous curve. Figure 80 represents the graphs of the functions

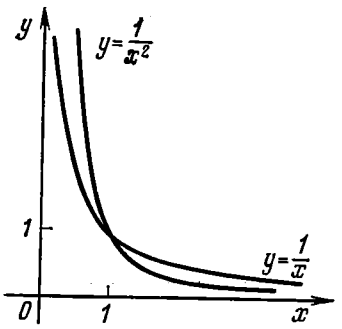


Fig. 80

$$y=\frac{1}{x}; \quad y=\frac{1}{x^3}$$

(for positive x's).

If $n=2m-1$ is an odd natural number, then the function

$$y=\frac{1}{x^{2m-1}}$$

is odd. The property of oddness implies that its graph is symmetric with respect to the origin.

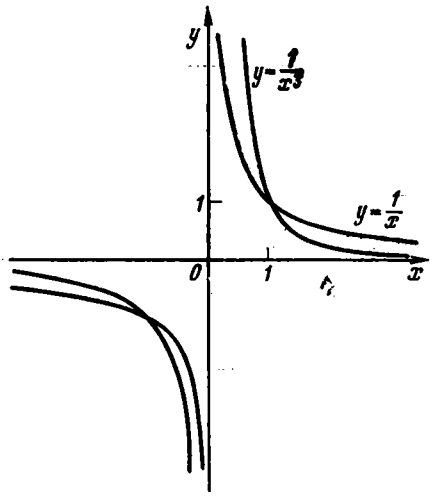


Fig. 81

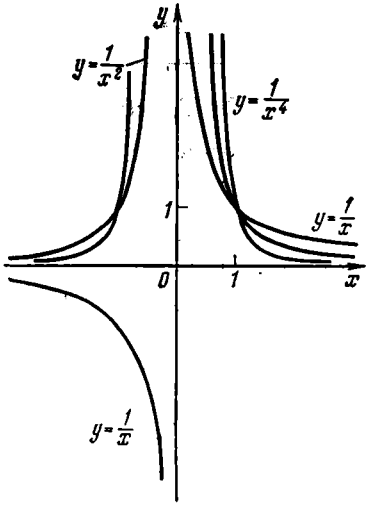


Fig. 82

Figure 81 represents the graphs of the functions

$$y=\frac{1}{x}; \quad y=\frac{1}{x^3}$$

(for all x's different from zero).

If $n = 2m$ is an even natural number, then the function

$$y = \frac{1}{x^{2m}}$$

is even. The evenness property implies that its graph is symmetric with respect to the y-axis.

Figure 82 represents the graphs of the functions

$$y = \frac{1}{x^2}; \quad y = \frac{1}{x^4}$$

and, for comparison, the graph of the function

$$y = \frac{1}{x}$$

(for all x's different from zero).

SECTION TEST

EXERCISES

1. Construct the graphs of the functions $y = \frac{1}{x^2}$; $y = \frac{1}{x^3}$; $y = \frac{1}{x^4}$.

2. Prove that for negative x's the function $y = \frac{1}{x^2}$ increases.

3. Prove that for negative x's the function $y = \frac{1}{x^3}$ decreases.

4. Prove that for negative x's the function $y = \frac{1}{x^2}$ is continuous.

5. Indicate the intervals of the x-axis on which the following functions increase and decrease:

(a) $y = \frac{1}{x^2}$; (b) $y = \frac{1}{x^3}$; (c) $\frac{1}{x^4}$;

(d) $y = \frac{1}{x^5}$; (e) $y = \frac{1}{x^6}$.

6. Which of the functions listed in Exercise 5 are even and which are odd? What does it mean that a function is even (odd)? Mention the characteristic features of graphs of even (odd) functions.

7. For what values of x are the functions dealt with in Exercise 1 not defined?

Chapter 25

ARITHMETIC AND GEOMETRIC PROGRESSIONS

Let every natural number $n = 1; 2; \ldots$ be associated, by virtue of a certain rule, with the number x_n. Thereby we define a sequence of numbers

$$x_1; \quad x_2; \quad x_3; \quad \ldots; \quad x_n; \quad \ldots,$$

or, simply, a sequence $\{x_n\}$. The numbers

$$x_1;\quad x_2;\quad x_3;\quad \ldots;\quad x_n;\quad \ldots$$

are called the terms of this sequence. and the term x_n is called its nth term or the general term of the sequence.

Here are several examples of numerical sequences:

$$-1;\quad -2;\quad -3;\ldots;\quad -n;\ldots;$$

$$1;\quad \frac{1}{4};\quad \frac{1}{9};\quad \frac{1}{16};\ldots;\quad \frac{1}{n^2};\ldots;$$

$$1;\quad \frac{1}{4};\quad \frac{1}{27};\quad \frac{1}{256};\ldots;\quad \left(\frac{1}{n}\right)^n;\ldots;$$

$$-1;\quad \frac{1}{2};\quad -\frac{1}{3};\quad \frac{1}{4};\ldots;\quad \frac{(-1)^n}{n};\ldots.$$

Sec. 25.1. Arithmetic Progression

An *arithmetic progression* (or sequence) is defined as a sequence of numbers each term of which, beginning with the second, is equal to the sum of the preceding term and a constant number called the *common difference*, or simply the *difference* of the progression.

Thus, if the sequence of numbers $\{a_n\}$ is an arithmetic progression with the difference d, then for any natural n

$$a_{n+1} = a_n + d.$$

For instance, the sequences

$$1;\quad 2;\quad 3;\quad 4;\quad \ldots; \tag{1}$$

$$3;\quad 1;\quad -1;\quad -3;\quad \ldots \tag{2}$$

are arithmetic progressions. Arithmetic progression (1) has the difference $d=1$, while Progression (2) has $d=-2$. For any arithmetic progression $\{a_n\}$ its nth term a_n can be expressed in terms of its first term a_1 and the difference of this progression d with the aid of the formula

$$a_n = a_1 + (n-1)\,d; \tag{3}$$

this formula is called the *formula of the general term* of an arithmetic progression.

For small natural n's Formula (3) can be proved in the following way:

$$a_2 = a_1 + d;$$
$$a_3 = (a_1 + d) + d = a_1 + 2d;$$
$$a_4 = (a_1 + 2d) + d = a_1 + 3d;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

At the $(n-1)$th step of this reasoning we get Formula (3):

$$a_n = a_1 + (n-1)\,d.$$

For very large n's this reasoning is unrealizable, therefore when deriving Formula (3) we are satisfied by the method of complete induction.

The proof of Formula (3) by the method of complete induction is set forth below.

For $n=1$ Formula (3) is true since $a_1=a_1+0\cdot d$. Suppose now that it is true $n=k$, that is, suppose the following equality is true:

$$a_k=a_1+(k-1)\,d.$$

Then, obviously, it is also true for $n=k+1$, since

$$a_{k+1}=a_k+d=a_1+(k-1)\,d+d=a_1+[(k+1)-1]\,d.$$

Thereby, according to the principle of complete induction, we have proved the validity of Formula (3) for any natural number n.

Using Formula (3), it is easy to verify the validity of the following statement: for any arithmetic progression $\{a_n\}$ for $m+n=k+l$ the following equality holds true:

$$a_m+a_n=a_k+a_l. \tag{4}$$

Indeed,

$$\begin{aligned}a_m+a_n&=a_1+d\,(m-1)+a_1+d(n-1)=2a_1+d(m+n-2)\\&=2a_1+d\,(k+l-2)=a_1+(k-1)\,d\\&\quad+a_1+(l-1)\,d=a_k+a_l.\end{aligned}$$

The number equal to the sum of the first n terms of an arithmetic progression is denoted by S_n, that is,

$$S_n=a_1+a_2+\ldots+a_{n-1}+a_n.$$

The terms a_1 and a_n are called *extreme terms* of the sum S_n. Using Property (4), we get the following formula:

$$S_n=\frac{(a_1+a_n)}{2}\cdot n.$$

Indeed,

$$\begin{aligned}2S_n&=S_n+S_n=(a_1+a_2+\ldots+a_n)+(a_1+a_2+\ldots+a_n)\\&=(a_1+a_n)+(a_2+a_{n-1})+\ldots+(a_n+a_1)=(a_1+a_n)\,n.\end{aligned}$$

Thus, *the sum of the first n terms of an arithmetic progression is equal to the half-sum of the extreme terms of this sum by the number of its terms.*

The sum S_n of an arithmetic progression can, obviously, be expressed in terms of its first term and the difference:

$$S_n=\frac{2a_1+d\,(n-1)}{2}\cdot n.$$

SECTION TEST

QUESTIONS

1. What is an arithmetic progression?
2. What is the difference of an arithmetic progression?
3. What formula is used to find the nth term of an arithmetic progression?
4. What formula is used to find the sum of the first n terms of an arithmetic progression?

EXERCISES

1. Find the 5th term of the arithmetic progression $\{a_n\}$ if $a_1 = -50$ and $d = 1.2$.
2. Find the sum of the first 30 terms of the arithmetic progression $\{a_n\}$ if $a_1 = -2.5$ and $d = 3$.
3. Is the number 34 a term of the arithmetic progression

$$-47;\ -44;\ -41;\ -38;\ \ldots\ ?$$

If the answer is affirmative, then find its number.

Sec. 25.2. Geometric Progression

A *geometric progression* (or sequence) is defined as a sequence of numbers every term of which, beginning with the second, is equal to its predecessor multiplied by a nonzero number which is constant for a given sequence of numbers. This number is called the *common ratio* (or, simply, the *ratio*).

Thus, if the sequence $\{a_n\}$ is a geometric progression with the ratio q, then $q \neq 0$ and for any natural n

$$a_{n+1} = a_n \cdot q.$$

For example, the sequences

$$2;\ 4;\ 8;\ 16;\ 32;\ \ldots;$$
$$1;\ -1;\ 1;\ -1;\ 1;\ \ldots;$$
$$\frac{1}{7};\ -\frac{1}{21};\ \frac{1}{63};\ -\frac{1}{189};\ \ldots$$

are geometric progressions having the ratios 2, (-1), $\left(-\frac{1}{3}\right)$, respectively.

For any geometric progression $\{a_n\}$ its nth term a_n can be expressed in terms of its first term and the ratio of this progression with the aid of the formula

$$a_n = a_1 q^{n-1} \qquad (1)$$

Indeed,

$$a_2 = a_1 q;$$
$$a_3 = (a_1 q)\, q = a_1 q^2;$$
$$a_4 = (a_1 q^2)\, q = a_1 q^3;$$
$$a_5 = (a_1 q^3)\, q = a_1 q^4;$$
$$\cdots\cdots\cdots$$

and we have proved Formula (1) for $n = 2; 3; 4; 5; \ldots$.

Thus, reasoning consecutively, we can obtain Formula (1) for those n's for which the indicated successive considerations are practically realizable. For an arbitrary n there arises the necessity to apply the method of complete induction.

Let us prove Formula (1) using the method of complete induction.

For $n = 1$ Formula (1) is, obviously, true: $a_1 = a_1 q^0$. Suppose it is true for $n = k$, that is, let us assume that

$$a_k = a_1 q^{k-1}.$$

Then it is also true for $n = k + 1$, since

$$a_{k+1} = a_k q = (a_1 q^{k-1})\, q = a_1 q^{(k+1)-1}.$$

Thereby, using the principle of complete induction, we have proved Formula (1) for any natural number n.

The sum

$$S_n = a_1 + a_2 + \ldots + a_n$$

of the first n terms of the geometric progression $\{a_n\}$ *with the ratio* q *is equal to*

$$S_n = na_1 \quad \text{for } q = 1, \tag{2}$$

$$S_n = \frac{a_1(1-q^n)}{1-q} = \frac{a_1 - a_n q}{1-q} \quad \text{for } q \neq 1. \tag{3}$$

Proof. For $q = 1$ Formula (2) is obvious. Let now $q \neq 1$. The following obvious equalities are true:

$$S_n(1-q) = S_n - S_n q$$
$$= a_1 + a_1 q + \ldots + a_1 q^{n-1} - a_1 q - a_1 q^2 - \ldots - a_1 q^n$$
$$= a_1 - a_1 q^n = a_1(1-q^n).$$

Consequently,

$$S_n(1-q) = a_1(1-q^n),$$

and since $q \neq 1$, we have

$$S_n = \frac{a_1(1-q^n)}{1-q} = \frac{a_1 - a_n q}{1-q}.$$

SECTION TEST

QUESTIONS

1. What is a geometric progression?
2. What is the ratio of a geometric progression?
3. What formula is used to find the nth term of a geometric progression?
4. What formula is used to find the sum of first n terms of a geometric progression?

EXERCISES

1. Find the 15th term of the geometric progression $\{a_n\}$ if $a_1 = -0.001$ and $q = 10$.
2. Given: the 8th term and the ratio of the geometric progression $\{a_n\}$: $a_8 = 2.56$ and $q = 2$. Find the sum of the first 16 terms of the progression.

Sec. 25.3. A Worked Problem

An odd number of stones lay along the road spaced at a distance of 10 m. These stones should be gathered at the place where the middle stone was situated. A man can carry only one stone, therefore he carried them one after the other, beginning with one of the extreme stones. To gather all the stones, the man covered a path of 3 km. How many stones were there?

Solution. Suppose there were $n = 2k + 1$ stones (since n is an odd number). Then there were k stones to the right and k stones to the left of the middle stone. Suppose the man began with the extreme right stone. To bring it to the middle stone, he had to cover the distance equal to $10 \cdot k$ metres. To take the next stone, the man had first to return to the place where it lay, that is, to walk $10 \cdot (k - 1)$ metres and, on taking the stone, to cover the distance $10 \cdot (k - 1)$ metres once again, that is, to fetch the second stone, he had to cover a path of $2 \cdot 10 (k - 1)$ metres. To fetch the third stone, he had to walk $2 \cdot 10 (k - 2)$ metres, and so forth. Finally, to fetch the kth stone, he had to walk $2 \cdot 10 \cdot 1$ metres.

Hence, to gather all stones from the right-hand side of the middle stone, the man had to cover the path (in metres)

$$S_1 = 10k + 20 (k - 1) + 20 (k - 2) + \ldots + 20 \cdot 1,$$

this path being, obviously, independent of the order in which the man gathered the stones.

To pick up all the stones on the left-hand side of the middle stone, the man had first to cover the way to the extreme stone, that is, a path of $10 \cdot k$ metres, and then to repeat the path covered in gathering the stones on the right-hand side, that is, the path equal to S_1 metres. Hence, to gather all the stones on the left of the middle

stone, the man had to cover the path (in metres)

$$S_2 = S_1 + 10k.$$

Since the entire path covered by the man is equal to 3000 metres, we have

$$3000 = S_1 + S_2.$$

Thus, we have obtained the following system of three equations in three unknowns S_1, S_2, and k:

$$\begin{cases} S_1 = 10k + 20[(k-1) + (k-2) + \ldots + 1], \\ S_2 = S_1 + 10k, \\ S_1 + S_2 = 3000, \end{cases} \tag{1}$$

from which we have to find only k.

Substituting the expressions for S_1 and S_2 into the third equation of System (1), we get the equation

$$3000 - 30k = 40[(k-1) + \ldots + 1]. \tag{2}$$

Applying to the sum in the brackets the formula of the sum of $(k-1)$ terms of an arithmetic progression, we get

$$(k-1) + (k-2) + \ldots + 1 = \frac{(k-1+1)(k-1)}{2} = \frac{k(k-1)}{2}.$$

Then Equation (2) will be rewritten in the form

$$3000 - 30k = 20k(k-1),$$

or in the form

$$2k^2 + k - 300 = 0. \tag{3}$$

The discriminant of quadratic equation (3)

$$D = b^2 - 4ac = 1^2 - 4 \cdot 2 \cdot (-300) = 2401 > 0,$$

consequently, Equation (3) has two distinct roots:

$$k_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm 49}{4},$$

that is,

$$k_1 = 12, \quad k_2 = -\frac{25}{2}.$$

It is clear that only $k_1 = 12$ satisfies the conditions of the problem But then $n = 25$.

Consequently, there were 25 stones.

SECTION TEST

EXERCISES

1. A cyclist left the point A; an hour later a motorcyclist started in the same direction, and in another half an hour a car followed him, each of them driving with a constant speed. The car overtook the cyclist in 15 minutes after it had surpassed the motorcyclist, and the latter overtook the cyclist at a point 120 km distant from the point A. Find the speeds of the cyclist, motorcyclist, and the car if it is known that they form an arithmetic progression.

2. An old problem. When asked about the reward he wanted to get for his invention the inventor of the chessboard suggested to be given one grain of wheat for the first square of chessboard, two grains for the second square, four grains for the third square, and so on, just doubling the amount of grains for each subsequent square. How many grains would they have to give the inventor for all the 64 squares of his chessboard?

Chapter 26

BINARY NUMBER SYSTEM

Sec. 26.1. The Concept of Binary Number System

When recording numbers we usually make use of the decimal system. The basis of the decimal number system are the numerals (or digits) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. At the same time they are numbers: zero and the first nine natural numbers, the subsequent (after 9) natural numbers being represented as combinations of the indicated numerals. We know very well how this is done, and there is no need to recall this in more detail.

Note, for instance, that

$$328 = 3 \cdot 10^2 + 2 \cdot 10^1 + 8;$$
$$308 = 3 \cdot 10^2 + 0 \cdot 10^1 + 8;$$
$$1001 = 1 \cdot 10^3 + 0 \cdot 10^2 + 0 \cdot 10^1 + 1.$$

But natural numbers may be represented in the binary system using a basic group of two digits 0 and 1. These digits are at the same time numbers. Binary notations of natural numbers following 1 consist only of the digits 0 and 1. How this looks like will be seen from the below equalities. In each of them, the first and second members are represented in decimal notation, while the third member in binary notation:

$$0 = 0 = 0;$$
$$1 = 1 = 1;$$
$$2 = 1 \cdot 2 + 0 = 10;$$
$$3 = 1 \cdot 2 + 1 = 11;$$

$$4 = 1 \cdot 2^2 + 0 \cdot 2 + 0 = 100;$$
$$5 = 1 \cdot 2^2 + 0 \cdot 2 + 1 = 101;$$
$$6 = 1 \cdot 2^2 + 1 \cdot 2 + 0 = 110;$$
$$7 = 1 \cdot 2^2 + 1 \cdot 2 + 1 = 111;$$
$$8 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 0 = 1000;$$
$$9 = 1 \cdot 2^3 + 0 \cdot 2^4 + 0 \cdot 2 + 1 = 1001;$$
$$10 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 0 = 1010;$$
$$11 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 = 1011,$$
$$12 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 0 = 1100;$$
$$13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1 = 1101;$$
$$14 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 0 = 1110;$$
$$15 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 1 = 1111;$$
$$16 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 0 = 10\,000;$$
$$17 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 1 = 10\,001.$$

We see that any natural number can be represented in binary notation with the aid of the digits 0 and 1 in a unique way, as it will be seen later. Of course, in common life the binary system is inconvenient for representing natural numbers. Even small natural numbers look rather awkward in binary notation. For example, 9 represented in binary notation is a four-digit number, and 17 is a five-digit number.

But it is of interest to note that modern digital computers generally do arithmetical operations in the binary system since it is more convenient for them as compared with the decimal system. They convert the numbers in decimal representation to equal numbers represented in the binary system, do all necessary operations, and convert the results again to decimal notation.

Let there be given a natural number in binary notation, for instance,

100 101.

Its last digit is said to be the digit *of the first place*, the last but one digit the digit *of the second place*, its predecessor the digit *of the third place*, and so forth.

For example, for the given number 100 101 the digits of its places (positions) are seen from the following table:

Place	...	6	5	4	3	2	1
Digit	...	1	0	0	1	0	1

In general, for a natural number represented in binary notation the *digit of the kth place* (*position*) is defined as the digit written in the kth place if the digits are counted from right to left.

The above examples illustrate the process of changing a natural number from binary notation to decimal notation. Here are some more examples:

$$100\,101 = 1\cdot 2^5 + 0\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2 + 1 = 32 + 4 + 1 = 37;$$

$$11\,100 = 2^4 + 2^3 + 2^2 = 16 + 8 + 4 = 28.$$

The reverse process, that is, conversion from decimal to binary, is more complicated. The relevant technique will be shown in the next section by way of examples.

SECTION TEST

QUESTIONS

1. What is the binary number system?
2. What is called the digit of the kth place of a natural number represented in binary notation?
3. How is a number given in binary notation converted to decimal notation?

EXERCISES

1. Give the binary representation of the following numbers:
(a) 2; (b) 2^2; (c) 2^3; (d) 2^4; (e) 2^5.
2. Convert the following numbers from binary to decimal:
(a) 101; (b) 100 000; (c) 101 000; (d) 111.

Sec. 26.2. Converting Numbers from Decimal to Binary

The below examples show how to change numbers represented in decimal notation to binary notation.

Example 1. Change the number 327 represented in decimal notation to binary notation.

Consider the numbers

$$1;\quad 2;\quad 2^2;\quad 2^3;\quad 2^4;\quad \ldots$$

and choose a natural number n for which

$$2^n \leqslant 327 < 2^{n+1}.$$

Such a number n, obviously, exists and is unique.

Indeed,

$$256 = 2^8 \leqslant 327 < 2^9 = 512,$$

therefore $n = 8$. This shows that in binary notation the number 327 is nine-digit. Let us compile a table (it is given below). For the ninth place put 1.

Now we compute the difference

$$327 - 2^8 = 327 - 256 = 71$$

and find for it the number n for which

$$2 \leqslant 71 < 2^{n+1}.$$

Since

$$64 = 2^6 \leqslant 71 < 2^7 = 128,$$

$n = 6$. This shows that the digit occupying the seventh place in our number represented in binary notation is 1, and the digit occupying the eighth place is 0. We write these digits into the table.

Further,

$$71 - 2^6 = 71 - 64 = 7$$

and

$$4 = 2^2 \leqslant 7 < 2^3 = 8.$$

This shows that the digit of the third place is 1, the digits of the fourth, fifth, and sixth places being 0. We enter them into the table. Finally,

$$7 - 2^2 = 3; \quad 2 = 2^1 \leqslant 3 < 2^2 = 4.$$

Consequently, the digit of the second place is 1 and the remainder

$$3 - 2 = 1,$$

that is, the digit of the first place is 1. We enter them into the table:

Place	...	9	8	7	6	5	4	3	2	1
Digit	...	1	0	1	0	0	0	1	1	1

Thus, we get

$$327 = 101\,000\,111.$$

The left-hand side of this equality is the given number in the decimal representation, its right-hand side being the same number in binary notation. This notation is unique since at any step of our reasoning the corresponding n was unique.

Example 2. Change the number 831 to binary notation.

We have

$$512 = 2^9 \leqslant 831 < 2^{10} = 1024,$$

that is, $n = 9$, consequently, in binary notation the number 831 is ten-digit; its digit of the tenth place being 1.

Further,

$$831 - 2^9 = 831 - 512 = 319;$$

$$256 = 2^8 \leqslant 319 < 2^9 = 512,$$

consequently, in the binary system the digit of the ninth place is 1. Since

$$319 - 2^8 = 319 - 256 = 63$$

and

$$32 = 2^5 \leqslant 63 < 2^6 = 64,$$

the digit occupying the sixth place (position) is 1, the digits of the eighth and seventh places being zeros.

Further

$$63 - 2^5 = 63 - 32 = 31$$

and

$$16 = 2^4 \leqslant 31 < 2^5 = 32,$$

consequently, the digit of the fifth place is 1.

Since

$$31 - 2^4 = 31 - 16 = 15$$

and

$$8 = 2^3 \leqslant 15 < 2^4 = 16,$$

the digit of the fourth place is 1.

Since

$$15 - 2^3 = 15 - 8 = 7 \text{ and } 4 = 2^2 \leqslant 7 < 2^3 = 8$$

the digit of the third place is 1.

Further,

$$7 - 2^2 = 7 - 4 = 3$$

and

$$2^1 \leqslant 3 < 2^2 = 4,$$

consequently, the digit of the second place turns out to be 1

Finally,

$$3 - 2 = 1,$$

consequently, the digit of the first place is 1.

The appropriate table has the form:

Place	...	10	9	8	7	6	5	4	3	2	1
Digit	...	1	1	0	0	1	1	1	1	1	1

We have obtained

$$831 = 1\,100\,111\,111,$$

where the right-hand side is the binary representation of the number 831.

SECTION TEST

EXERCISES

1. Convert the following numbers from binary to decimal notation: (a) 23; (b) 62; (c) 232; (d) 421.

Sec. 26.3. Operations with Numbers in Binary Representation

Numbers given in binary notation are compared much in the same way as this is done in the decimal system. Arithmetic operations are also performed in a similar manner.

The following examples are clear without explanations:

$$101 < 1000;$$
$$101\,011 < 101\,100;$$
$$10\,001 < 10\,101.$$

When adding in the binary system, the following should be borne in mind:

$$0 + 0 = 0;$$
$$0 + 1 = 1 + 0 = 1;$$
$$1 + 1 = 10.$$

One should be used to the last equality, its right-hand side is the number 2 represented in binary notation.

Judging by these equalities, we arrive at the conclusion that the rules for adding numbers in the binary system do not differ from the corresponding rules in the decimal system.

Example 1.

$$\begin{array}{r} 1\,001 \\ +\ \ 110 \\ \hline 1\,111 \end{array}.$$

Example 2.

$$\begin{array}{r} 1\,001 \\ +\ \ 101 \\ \hline 1\,110 \end{array}.$$

Example 3.

$$\begin{array}{r} 1\,101 \\ +\ 1\,001 \\ \hline 10\,110 \end{array}.$$

Example 4.

$$\begin{array}{r} 110 \\ +\ 111 \\ \hline 1\,101 \end{array}.$$

When subtracting in the binary system, bear in mind that

$$\begin{aligned} 0 - 0 &= 0; \\ 1 - 0 &= 1; \\ 1 - 1 &= 0; \\ 10 - 1 &= 1. \end{aligned}$$

Thus, we see that subtraction in the "column" manner in the binary system is performed by the same rules as this is done in the decimal system.

Example 5.

$$\begin{array}{r} 1\,101 \\ -\ 1\,001 \\ \hline 100 \end{array}.$$

Example 6.

$$\begin{array}{r} 1\,001 \\ -\ 101 \\ \hline 100 \end{array}.$$

Example 7.

$$\begin{array}{r} 10\,101 \\ -\ 1\,111 \\ \hline 110 \end{array}.$$

In the binary number system the basic table of multiplication is very simple:

$$\begin{aligned} 0 \cdot 0 &= 0; \\ 0 \cdot 1 &= 1 \cdot 0 = 0; \\ 1 \cdot 1 &= 1. \end{aligned}$$

That is why multiplication is reduced to addition.

Example 8.

$$\begin{array}{r} 10\,101 \\ \times\ \ 11 \\ \hline 10\,101 \\ +\ 10\,101 \\ \hline 111\,111 \end{array}.$$

Example 9.

$$\begin{array}{r} \times \quad 10\,111 \\ \hline 101 \\ \hline 10\,111 \\ + \quad 10\,111 \\ \hline 1\,110\,011 \end{array}.$$

Here, the reader should take into consideration that $1 + 1 = 10$.

SECTION TEST

QUESTIONS

1. What are the basic rules for addition in the binary system?
2. What are the basic rules for multiplication in the binary system?

EXERCISES

1. Find the sum of the numbers represented in binary notation:
(a) $1010 + 111 + 1011$; (b) $1 + 1 + 1$;
(c) $1 + 1 + 1 + 1$; (d) $1 + 1 + 1 + 1 + 1$.
2. Find the difference between the numbers represented in binary notation:
(a) $101 - 11$; (b) $111 - 101$; (c) $10\,101 - 111$.
3. Find the product of the numbers given in binary notation:
(a) $101 \cdot 11$; (b) $111 \cdot 101$; (c) $1010 \cdot 11$.

Chapter 27

APPROXIMATE CALCULATIONS

Sec. 27.1. The Absolute Value of a Number

Recall that the absolute value of the number a is the number $|a|$ equal to a if a is nonnegative, and equal to $-a$ if a is negative, that is:

$$|a| = \begin{cases} a, & \text{if } a \geqslant 0, \\ -a, & \text{if } a < 0. \end{cases}$$

For instance,

$$|12| = 12; \quad |-12| = 12.$$

The absolute value of the number a is a nonnegative number. More precisely, the absolute value of the number a, different from zero, is a positive number, and the absolute value of zero is zero.

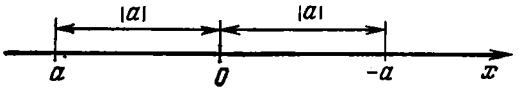


Fig. 83

If we mark the points a and $-a$ on the number axis (Fig. 83), then the absolute value $|a|$ expresses, obviously, the distance from either

of these points to the zero point (in Fig. 83 a is a negative number).

Basic properties of absolute values are considered below.

(1) *The absolute value of the product of two numbers is equal to the product of absolute values of these numbers*

$$|ab| = |a| \cdot |b|.$$

For instance,

$$|7 \cdot 3| = 7 \cdot 3 = |7| \cdot |3|;$$
$$|(-7) \cdot 3| = |-7 \cdot 3| = 7 \cdot 3 = |-7| \cdot |3|;$$
$$|(-7) \cdot (-3)| = |7 \cdot 3| = 7 \cdot 3 = |-7| \cdot |-3|.$$

(2) *The absolute value of the quotient of two numbers is equal to the quotient of the absolute values of these numbers*:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad (b \neq 0)$$

For example,

$$\left|\frac{7}{3}\right| = \frac{7}{3} = \frac{|7|}{|3|}; \quad \left|\frac{-7}{3}\right| = \left|-\frac{7}{3}\right| = \left|\frac{7}{3}\right| = \frac{|-7|}{|3|}; \quad \frac{|-7|}{|-3|} = \frac{7}{3} = \frac{|-7|}{|-3|}.$$

(3) *The absolute value of the sum of two numbers does not exceed the sum of the absolute values of these numbers*:

$$|a + b| \leqslant |a| + |b|.$$

Indeed, if both a and b are nonnegative, then

$$a + b = +(|a| + |b|);$$

and if both of them are nonpositive, then

$$a + b = -(|a| + |b|).$$

But then

$$|a + b| = |\pm (|a| + |b|)| = |a| + |b|.$$

If now the numbers a and b have unlike signs, and for definiteness $|a| \geqslant |b|$, then

$$a + b = +(|a| - |b|), \text{ if } a \text{ is positive,}$$

and

$$a + b = -(|a| - |b|), \text{ if } a \text{ is negative.}$$

But then

$$|a + b| = |\pm (|a| - |b|)| = |a| - |b| \leqslant |a| + |b|,$$

since

$$-|b| \leqslant |b|.$$

Thus, for any real numbers a and b

$$|a + b| \leqslant |a| + |b|.$$

(4) *The absolute value of the difference between two numbers does not exceed the sum of the absolute values of these numbers*:

$$|a - b| \leqslant |a| + |b|.$$

This property readily follows from Property (3). Indeed,

$$|a - b| = |a + (-b)| \leqslant |a| + |-b| = |a| + |b|.$$

SECTION TEST

QUESTIONS

1. What is the absolute value of a number?
2. What are the basic properties of the absolute values of numbers?

EXERCISES

1. Show that if $a < c < b$, then $|c - a| \leqslant b - a$ and $|b - a| < b - a$ (that is, if the point c of the number axis is situated between the points a and b, then the distance from c to a, the same as the distance from c to b, is less than the distance between a and b).
2. Show that if $a \leqslant c < b$, then $|c - a| < b - a$ and $|b - c| \leqslant b - a$

Sec. 27.2. Approximate Equality

If the number $\bar{a}$ differs but little from the number a, then we say that a is approximately equal to $\bar{a}$ and write

$$a \cong \bar{a}$$

We also say that $\bar{a}$ is an *approximation* to the number a. The symbol $\cong$ is the sign of approximate equality.

If for some positive number ε the following inequality is fulfilled

$$|a - \bar{a}| \leqslant \varepsilon$$

then a is said to be approximately equal to $\bar{a}$ with an accuracy to ε.

The number ε is called the *error of approximating* the number a with the aid of the number $\bar{a}$. It is clear that if ε is an aproximation error, then any number ε_1, exceeding it, ($\varepsilon_1 > \varepsilon$) is also an approximation error.

The number ε_0 equal to the absolute value of the difference $\bar{a} - a$, that is,

$$\varepsilon_0 = |\bar{a} - a|,$$

is the least approximation error. The number ε_0 will be called the *absolute error of approximation* to the number a with the aid of the number $\bar{a}$.

It is important to know the approximation error. But in practice it is by far not always can be known.

Therefore only the approximation error is usually found. And we try to represent the error in a sufficiently simple form and make sure that it is not very overstated.

If the number a satisfies the inequalities

$$a_1 \leqslant a \leqslant a_2,$$

and we regard that a_1 and a_2 are approximations to the number a ($a_1 \cong a$, $a_2 \cong a$), then a_1 is said to be an approximation with deficit (from below), and a_2 an approximation with excess (from above) to the number a with an accuracy to $a_2 - a_1$.

Example 1. There is an approximate equality

$$17.32 \cong 17$$

with an accuracy to 1, since

$$| 17.32 - 17 | = 0.32 < 1.$$

In this case the absolute error is equal to 0.32, but we deemed it necessary to take the number 1 as an error.

For instance, if the distance between two bus stops is equal to 17.32 km, then in many cases we would say that it is equal to 17 km and we would be understood in the sense that 17 km is an approximate value of the distance between the stops with an accuracy to 1 km.

Example 2. We have the approximate equality

$$879 \cong 900$$

with an accuracy to 50, since

$$| 900 - 879 | = 21 < 50.$$

In the present case the absolute error of approximation is equal to 21, but we deemed it necessary to take the number 50 as an error. For example, when selling railway tickets, they express long distances approximately with an accuracy to 50 km.

Example 3. When measuring temperature of a patient, the level of the mercury column of the thermometer was found to occupy the position between 38.3° and 38.4°. Let T be the true temperature of the patient. Then we say that T is approximately equal to 38.3° to within 0.1° with deficit, or T is approximately equal to 38.4° to within 0.1° with excess and write

$$T \cong 38.3°; \quad T \cong 38.4°. \tag{1}$$

We do not know the exact value of T, the absolute error of the indicated approximations is also unknown. But we do know that

$$38.3° < T < 38.4°.$$

Hence it follows that

$$| T - 38.3° | < 0.1; \quad | T - 38.4° | < 0.1.$$

Therefore the approximations (1) are accurate to within 0.1.

Example 4. For the number

$$a = -13.23089107$$

the following inequalities take place:

$$-13.230 > a > -13.231,$$

which show that

$$-13.231 \cong a \text{ and } -13.230 \cong a$$

with an accuracy to 0.001 with deficit and excess, respectively.

Example 5. For the number

$$a = 0.02\,(372)$$

we have the inequalities

$$0.0237 < a < 0.0238,$$

showing that

$$0.0237 \cong a \text{ and } 0.0238 \cong a$$

to within 0.0001 with deficit and excess, respectively.

Decimals can be approximated by *rounding off*. Let us explain the process of rounding off, taking as an example the number

$$a = 37.57891.$$

From the inequalities

$$37.578 < a < 37.579$$

it follows that the numbers 37.578 and 37.579 approximate a to within 0.001 with deficit and excess, respectively. When rounding off, the nearest to a number a is taken as an approximation to a. Thus, we obtain an approximate equality rounded off to the third decimal place:

$$37.579 \cong a.$$

The error of this approximation is, obviously, equal to

$$\frac{1}{2} \cdot 0.001 = 0.0005.$$

Here is another example of rounding off a decimal fraction:

$$a = -5.317.$$

Since

$$-5.32 < a < -5.31$$

and the number -5.32 is nearest to a, we have

$$a \cong -5.32,$$

and the error of this approximation is equal to

$$\frac{1}{2} \cdot 0.01 = 0.005.$$

Whole numbers are also rounded off. For example

$$-3789 \cong -3800; \quad 279{,}189 \cong 279{,}000.$$

The error of the first approximation is equal to 50, of the second to 500.

SECTION TEST

QUESTIONS

1. What is an approximate equality?
2. What is an error and absolute error of approximation?
3. What is an approximate equality with rounding off?
4. If in the process of approximating a given decimal by rounding off the first four digits to the right of the decimal point are retained and the remaining digits are replaced by zeros, then what error will be obtained?
5. How many digits after the decimal point must be retained in a decimal fraction with the remaining digits replaced by zeros in order for the obtained number to approximate the given fraction with an accuracy to 1/20; 1/35; 1/200; 1/375?

EXERCISES

1. Write the approximations with deficit and excess to within 0.001 for the following numbers:
(a) 37.57891; (b) 0.002576; (c) -117.00992; (d) 0.3(9); (e) $-31.72(13)$; (f) 0.00(03); (g) 32.10(004); (h) -2.378523.
2. Write the approximations with rounding off for the numbers indicated in Exercise 1 leaving three digits after the decimal point and replacing the remaining digits by zeros.

Sec. 27.3. The Absolute Error of Approximation

Let u denote a number and $\bar{u}$ its approximation ($u \cong \bar{u}$). The difference between $\bar{u}$ and u will be denoted by Δu:

$$\bar{u} - u = \Delta u$$

(Δ the Greak letter "delta"). The expression Δu is read: "delta u or increment u". Since

$$\bar{u} = u + \Delta u$$

in order to obtain from the number u its approximate value (approximation) we have to add to it the increment Δu.

The absolute value of Δu, that is,

$$|\Delta u| = |\bar{u} - u|,$$

was called (in the preceding section) the absolute error or the absolute error of the approximation to u. And any number ε, not less than the absolute error, was called the error of approximation ($\varepsilon \geqslant |\Delta u|$).

We often know about the number u that it exists, but do not know what it is exactly equal to. But we do know its approximation $\bar{u}$ and the error of approximation, that is, a certain positive number ε for which

$$|\Delta u| \leqslant \varepsilon.$$

Therefore we say that $u \cong \bar{u}$ with an accuracy to ε.

For example, the temperature of the patient at a given moment is a definite number T, whose exact value, however, we do not know. But looking at the thermometer we find out its approximate value $\bar{T}$ with an error less than 0.1°:

$$|T - \bar{T}| = |\Delta T| < 0.1°.$$

Let there be given numbers u and v, and let $\bar{u}$ and $\bar{v}$ be their approximations. We shall regard that the sum $\bar{u} + \bar{v}$ and the difference $\bar{u} - \bar{v}$ approximate the sum $u + v$ and the difference $u - v$, respectively, that is, we shall consider the approximations

$$u + v \cong \bar{u} + \bar{v}, \quad u - v = \bar{u} - \bar{v}.$$

Let us estimate the absolute errors of these approximations:

$$|\Delta(u+v)| = (\bar{u}+\bar{v}) - (u+v)| = |(\bar{u}-u) + (\bar{v}-v)|$$
$$\leqslant |\bar{u}-u| + |\bar{v}-v| = |\Delta u| + |\Delta v|,$$

that is,

$$|\Delta(u+v)| \leqslant |\Delta u| + |\Delta v|.$$

The left-hand side of this inequality represents the absolute error of approximation of the sum of the numbers u and v, while the right-hand side the sum of the absolute errors of approximations to these numbers.

Likewise,

$$|\Delta(u-v)| = |(\bar{u}-\bar{v}) - (u-v)|$$
$$= |(\bar{u}-u) - (\bar{v}-v)| \leqslant |\bar{u}-u| + |\bar{v}-v| = |\Delta u| + |\Delta v|,$$

that is,

$$|\Delta(u-v)| \leqslant |\Delta u| + |\Delta v|.$$

Now, the left-hand side of the inequality is the absolute error of approximation of the difference between the numbers u and v, and the right-hand side the sum of the absolute errors of these numbers.

Thus, we have arrived at the following statement:

The absolute error of approximation to the sum or difference of two numbers does not exceed the sum of absolute errors of approximations to these numbers.

Using this statement, it is easy to find the error of approximation to the sum (or difference) of two numbers.

Example 1. The approximate equality

$$3.47586 - 2.3697 \cong 3.47 - 2.36$$

takes place with an accuracy to

$$2 \cdot \frac{1}{100} = \frac{1}{50},$$

and the approximate equality (rounded off)

$$3.47586 - 2.3697 \cong 3.48 - 2.37$$

with an accuracy to

$$2 \cdot \frac{10^{-2}}{2} = \frac{1}{100}.$$

Example 2. The approximate equality

$$3.7235 + (-5.0123) \cong 3.72 + (-5.01)$$

takes place with an accuracy to

$$2 \cdot \frac{10^{-2}}{2} = \frac{1}{100},$$

since it is possible to regard that we approximated the terms by rounding off.

Example 3. The approximate equality

$$-3.(01) - [-6.(724)] \cong 3.010 - (-6.724)$$

takes place with an accuracy to

$$2 \cdot 0.001 = 0.002.$$

SECTION TEST

QUESTIONS

1. What is the absolute error of approximation?

2. How do we estimate the absolute error of approximation to the sum (or difference) of two numbers?

EXERCISES

1. Given two numbers:
(a) $\alpha = 12.35817$, $\beta = 6.9879$; (d) $\alpha = 3.12(27)$, $\beta = 1.22(891)$;
(b) $\alpha = 7.1723$, $\beta = 0.8192$; (e) $\alpha = 17.23(38)$, $\beta = -21.(136)$.
(c) $\alpha = 11.1429$, $\beta = 3.2872$;
Find approximately the sum and difference of these numbers by replacing their digits beginning with the fourth decimal place.
What will be the error of approximation?

2. How many digits must be retained in the decimals considered in Exercise 1 (the rest of the digits being replaced by zeros) so that the error of their addition or subtraction be less than 1/30; 1/350; 1/4200?

3. Find approximately the sum and difference of the following two numbers α and β bearing in mind that the final result must be obtained with an accuracy to 0.01:
(a) $\alpha = 3.1567$, $\beta = 2.0921$;
(b) $\alpha = 17.3281$, $\beta = -2.9856$;
(c) $\alpha = -7.0003$, $\beta = -2.9812$.

Sec. 27.4. The Absolute Error of Approximation to the Sum of Several Terms

The statement on the approximation error for the sum of two terms is generalized for any finite number of terms. Namely, the following statement takes place: the absolute error of approximation to the sum of a finite number of terms does not exceed the sum of the absolute errors of approximations to these terms.

Indeed, let there be given the approximate equalities

$$u \cong \bar{u}, \quad v \cong \bar{v}, \ldots, w \cong \bar{w},$$

$$u+v+\ldots+w \cong \bar{u}+\bar{v}+\ldots+\bar{w}.$$

Then

$$\begin{aligned} |\Delta(u+v+\ldots+w)| &= |(\bar{u}+\bar{v}+\ldots+\bar{w})-(u+v+\ldots+w)| \\ &= |(\bar{u}-u)+(\bar{v}-v)+\ldots+(\bar{w}-w)| \\ &= |\Delta u+\Delta v+\ldots+\Delta w| \leqslant |\Delta u|+|\Delta v|+\ldots+|\Delta w|, \end{aligned}$$

that is,

$$|\Delta(u+v+\ldots+w)| \leqslant |\Delta u|+|\Delta v|+\ldots+|\Delta w|.$$

The left-hand side of this inequality represents the absolute error of approximation to the sum of the terms $u, v, \ldots, w$, the right-hand side the sum of the absolute errors of approximations to these terms, which just proves the above statement.

Example 1. Given a table of approximations for twenty numbers:

$$u_1; \quad u_2; \ldots; \quad u_{20}.$$

The first ten of these approximations

$$\bar{u}_1; \quad \bar{u}_2; \ldots; \quad \bar{u}_{10}$$

are given within the third decimal place with rounding off. And the subsequent ten approximations

$$\bar{u}_{11};\quad \bar{u}_{12};\ \ldots;\quad \bar{u}_{20}$$

are given with an accuracy to the second decimal place and also rounded off.

Let us find the error of the approximate equality

$$u_1+u_2+\ldots+u_{20}\cong \bar{u}_1+\bar{u}_2+\ldots+\bar{u}_{20}. \tag{1}$$

To this end, we have to add the errors of the approximate equalities:

$$u_1\cong \bar{u}_1;\ \ldots u_{10}\cong \bar{u}_{10};\quad u_{11}\cong \bar{u}_{11};\ \ldots u_{20}\cong \bar{u}_{20}.$$

Each of the first ten approximations has the error 1/2000, while each of the last ten approximations has the error 1/200.

Therefore the absolute error of approximate equality (1) does not exceed the number

$$10\cdot\frac{1}{2000}+10\cdot\frac{1}{200}=\frac{1}{200}+\frac{1}{20}=\frac{11}{200}.$$

Example 2. Given below is a table of twenty numbers (see the first column). It is required to compute approximately the sum of these numbers to within 0.1.

	To within the 3rd decimal place	To within the 2nd decimal place		To within the 3rd decimal place	To within the 2nd decimal place
3.7568	3.756	3.76	0.999	0.999	1.00
4.2112	4.211	4.21	2.11122	2.111	2.11
13.894	13.894	13.89	3.78904	3.789	3.79
14.72122	14.721	14.72	5.00702	5.007	5.01
2.0013	2.001	2.00	8.01213	8.012	8.01
3.00056	3.000	3.00	9.2466	9.246	9.25
2.10127	2.101	2.10	11.38910	11.389	11.39
3.3348	3.334	3.33	3.00002	3.000	3.00
5.6292	5.629	5.63	2.0012	2.001	2.00
0.01921	0.019	0.02	3.57892	3.578	3.58

If the numbers in the table are given with an accuracy to 10^{-k}, then their sum will differ from the true sum by no more than the quantity $\sigma = 20.10^{-k}$.

For $k = 1$ we get $\sigma = 2$; for $k = 2$, $\sigma = 0{,}2$; for $k = 3$, $\sigma = 0.02 < 0.1$.

Thus, the given numbers should be taken with three digits to the right of the decimal point. In such a form these numbers are written in the second column. If they are added together, then the obtained sum will be approximately equal to the true value to within 0.02, the more so to within 0.1.

Written in the third column are rounded off approximations to our numbers with two decimal places retained. Obviously, the numbers in the third column approximate the corresponding numbers with an accuracy to $\frac{1}{2}\cdot 10^{-2}$. But then their sum will be approximately equal to the true sum with an error equal to

$$20\cdot\frac{1}{2}\cdot 10^{-2}=0.1.$$

We see that the set problem may be solved more economically by adding the numbers of the third column.

SECTION TEST

QUESTIONS

1. How do we estimate the absolute error of approximating the sum of a finite number of terms?

EXERCISES

1. The floor of a room consists of 17 boards. Each of them is 35 cm wide with an accuracy to 1/2 cm. What possible error shall we take by considering that the dimension d of the room in the direction of the width of the boards is equal to $d \cong 35 \times 17$ cm?

2. Add approximately the indicated numbers so as to obtain the final result to within 0.1: 0.378561; 3.789012; 7.178210; 11.532478; 2.235622; 4.251617; 9.100017. Carry out all necessary simplifications.

3. In Exercise 2, in the table of numbers retain three decimal places and round off each number. Find out with what accuracy does the sum of the numbers thus obtained approximate the true sum.

Sec. 27.5. The Relative Error of Approximation

The *relative error* of the approximate equality

$$u \cong \bar{u}$$

is defined as the ratio of its absolute error to the absolute value o the number u, that is,

$$\frac{|\Delta u|}{|u|}=\left|\frac{\Delta u}{u}\right|.$$

For instance, if

$$u = 17.83, \quad \bar{u} = 18,$$

then we have the approximate equality

$$17.83 \cong 18$$

with the absolute error

$$|\Delta u| = 0.17$$

and relative error

$$\left|\frac{\Delta u}{u}\right| = \frac{0.17}{17.83} < \frac{0.17}{17} = 0.01,$$

which, as we see, is estimated from above by the number 0.01. If some number exceeds a given one, then it is said to *estimate* the given number *from above* or that it is the *upper bound* of the given number. Usually, as an upper bound, we choose a number of the form 10^k, where k is an integer (positive or negative).

If somebody measured the length of a workpiece whose true length is 10 cm and made an absolute error equal to 1 cm, then we would say that this is a bad measurement since the approximation is rough. However, if the length of a room whose true length equals 5 m was measured with the same absolute error of 1 cm, we would say that this measurement is not bad. As a rule, the accuracy of approximation is characterized not by the absolute, but by the relative error.

In measuring the workpiece the relative error was equal to 1/10, while in the example with the room the relative error was equal to 1/500. The less the relative error of approximation, the more exact the approximation.

In practice, we often have to simplify numbers by replacing in a whole number its last digits by zeros or by replacing in a decimal fraction all the digits to the right of the decimal point beginning with a certain one.

If it is known that there are 254,624 inhabitants in a town, then it is usually sufficient to say that there are 254,000 inhabitants in the town, or 250,000, or even 200,000.

The decimal fraction 0.037123 will be simplified if we replace its three last digits by zeros thus obtaining the decimal 0.037, or the four last digits, thus obtaining the decimal 0.03. However, in the last case it would be natural to round off the result, that is, to take the number 0.04.

In connection with these examples there arises a question of estimating the error in such simplifications which may be regarded as approximate equalities. It is convenient to obtain this estimate if we write a given positive number in scientific notation:

$$a \cdot 10^k$$

($a = \alpha_0, \alpha_1, \alpha_2, \ldots, 1 \leqslant a < 10$, k an integer).

Let us simplify this number by retaining s digits to the right of the decimal point in the decimal a. The obtained number will be denoted by $a^{(s)}$.

We obtain the approximate equality

$$a \cdot 10^k \cong a^{(s)} \cdot 10^k. \tag{1}$$

Obviously, its absolute error does not exceed

$$10^{-s} \cdot 10^k,$$

and the relative error does not exceed

$$\frac{10^{-s} \cdot 10^k}{a \cdot 10^k} \leqslant \frac{10^{-s} \cdot 10^k}{1 \cdot 10^k} = 10^{-s},$$

since $a \geqslant 1$.

Thus, we have justified the following rule.

If a given positive number is written in scientific notation $a \cdot 10^k$ $(1 \leqslant a < 10)$ and in its first multiplier (the number a) s digits are retained to the right of the decimal point, then the obtained approximate equality (1) will have the absolute error not exceeding $10^{-s} \times 10^k = 10^{k-s}$ and relative error not exceeding 10^{-s}.

Example 1. Given the approximate equality

$$357{,}281 \cong 357{,}000. \tag{2}$$

Writing both sides in scientific notation, we get

$$3.57281 \cdot 10^5 \cong 3.57 \cdot 10^5. \tag{3}$$

The right-hand side of approximate equality (3) has been obtained by retaining two digits to the right of the decimal point in the first multiplier of the left-hand member. But then, on the basis of the above rule, the absolute error of Approximation (3), and consequently, of Approximation (2), does not exceed $10^{-2} \cdot 10^5 = 10^3$. And the relative error does not exceed 10^{-2}.

Example 2. The approximate equality

$$0.002372 \cong 0.0023 \tag{4}$$

may be written in the following way:

$$2.372 \cdot 10^{-3} \cong 2.3 \cdot 10^{-3}.$$

Here both sides are already represented in scientific notation.

Since the first multiplier of the right-hand side of this approximate equality is obtained from the first multiplier of the left-hand side by retaining in the latter one digit to the right of the decimal point, the absolute error of approximation does not exceed $10^{-1} \cdot 10^{-3} = 10^{-4}$, while the relative error does not exceed $10^{-1} = 0.1$.

Recall that the relative error of approximation to a number is equal to its absolute error divided by its absolute value. This shows that the relative error of a negative number is equal to the relative error of a positive number, that is, of its absolute value.

SECTION TEST

QUESTIONS

1. What does it mean to estimate a given number from above?
2. What is the relative error of approximation?
3. What is the rule for estimating the absolute and relative errors in simplifying a number?

EXERCISES

1. Estimate the relative error for each of the following approximations:
(a) $23.392 \cong 23.000$; (b) $25.136 \cong 25$;
(c) $0.324 \cong 0.3$; (d) $0.000578 \cong 0.0006$.
2. Simplify the following numbers by replacing the last two digits by zeros:
(a) 71.523; (b) 0.568; (c) 0.00328.
Estimate the absolute and relative errors of the approximations thus obtained.

Sec. 27.6. Approximating a Product

Consider the approximate equalities

$$u \cong \bar{u}, \quad v \cong \bar{v}.$$

We shall assume that $u \neq 0$ and $v \neq 0$, and that the product $\bar{u}\bar{v}$ approximates the product uv:

$$uv \cong \bar{u}\,\bar{v}. \tag{1}$$

Since

$$\bar{u} = u + \Delta u, \quad \bar{v} = v + \Delta v,$$

the absolute error of Approximation (1) is estimated in the following way:

$$\begin{aligned} |\,\Delta(uv)\,| &= |\,\bar{u}\bar{v} - uv\,| \\ &= |\,(u+\Delta u)(v+\Delta v) - uv\,| = |\,v\,\Delta u + u\,\Delta v + \Delta u\,\Delta v\,| \\ &\leqslant |\,v\,|\,|\,\Delta u\,| + |\,u\,|\,|\,\Delta v\,| + |\,\Delta u\,|\,|\,\Delta v\,|, \end{aligned}$$

that is,

$$|\,\Delta(uv)\,| \leqslant |\,v\,|\,|\,\Delta u\,| + |\,u\,|\,|\,\Delta v\,| + |\,\Delta u\,|\,|\,\Delta v\,|. \tag{2}$$

Dividing Inequality (2) by the product $|\,u\,|\,|\,v\,|$ which is, by the supposition, not equal to zero, we get

$$\left|\frac{\Delta(uv)}{uv}\right| \leqslant \left|\frac{\Delta u}{u}\right| + \left|\frac{\Delta v}{v}\right| + \left|\frac{\Delta u}{u}\right|\left|\frac{\Delta v}{v}\right|. \tag{3}$$

Thus, the relative error of approximating the product of two numbers is estimated with the aid of Formula (3).

Approximations to numbers are usually chosen with small relative errors.

Then the product of the relative errors $\left|\frac{\Delta u}{u}\right|\left|\frac{\Delta v}{v}\right|$ is so small as compared with their sum $\left|\frac{\Delta u}{u}\right|+\left|\frac{\Delta v}{v}\right|$, that the product $\left|\frac{\Delta u}{u}\right|\left|\frac{\Delta v}{v}\right|$ in Formula (3) may often be neglected.

In such cases the relative error of approximating the product of two numbers is estimated by the inequality

$$\left|\frac{\Delta(uv)}{uv}\right| \leqslant \left|\frac{\Delta u}{u}\right|+\left|\frac{\Delta v}{v}\right|, \tag{4}$$

which shows that the *relative error of approximating the product of two numbers u and v does not exceed the sum of relative errors of approximating these numbers.*

Henceforth, we shall use Inequality (4) to estimate the relative error of approximating the product of two numbers.

Note that when approximately computing the product of the numbers u and v one should try to make the relative errors $\left|\frac{\Delta u}{u}\right|$ and $\left|\frac{\Delta v}{v}\right|$ equal.

Example 1. Find approximately the product of the numbers

$$u = 3.7586 \text{ and } v = 12.389$$

so that the result differs from the true product with the relative error less than 0.02.

Solution. Let us simplify the numbers u and v with relative errors not exceeding 0.01.

For this purpose (see the preceding section), we must reduce these numbers to scientific notation and retain the first two digits to the right of the decimal point in their first multipliers:

$$u = 3.7586 \cdot 1 \cong 3.75, \quad v = 1.2389 \cdot 10 = 1.23 \cdot 10.$$

We have obtained the approximate equalities

$$u \cong 3.75, \quad v \cong 1.23 \cdot 10 = 12.3$$

with relative errors not exceeding 0.01.

Since in the operation of multiplication the relative errors are added, the approximate equality

$$uv \cong 3.75 \cdot 12.3 = 46.125$$

takes place with a relative error not greater than 0.02.

Example 2. Calculate approximately the product of the numbers

$$a = 428.372 \quad \text{and} \quad b = 1.233$$

with a relative error less than 0.02. Estimate the absolute error of approximation.

Solution. Find the approximations to the numbers a and b with relative errors less than 0.01:

$$a = 428.372 = 4.28372 \cdot 10^2 \cong 4.28 \cdot 10^2 = 428,$$
$$b = 1.233 \cong 1.23 \cdot 1.$$

The obtained approximate equalities are taken with relative errors not exceeding 0.01. Therefore the approximate equality

$$ab \cong 428 \cdot 1.23 = 526.44$$

takes place with a relative error not exceeding 0.02.

SECTION TEST

QUESTIONS

1. How is the relative error of approximating the product of two numbers usually estimated?

EXERCISES

1. Compute approximately the product of the numbers

$$a = 0.123456789 \ldots \text{ and } b = 2.\overline{7}$$

with a relative error less than 0.001.

2. Compute approximately the product of the numbers

$$a = 7.9\overline{17} \text{ and } b = 1.2345678910$$

with a relative error less than 1/500.

Sec. 27.7. Approximating a Quotient

Let there be given the approximations to the numbers u and v, different from zero:

$$u \cong \bar{u}, \quad v \cong \bar{v}.$$

Let us estimate the relative error of approximating the quotient

$$\frac{u}{v} \cong \frac{\bar{u}}{\bar{v}}.$$

The absolute error is estimated in the following way:

$$\left|\Delta\left(\frac{u}{v}\right)\right| = \left|\frac{\bar{u}}{\bar{v}} - \frac{u}{v}\right| = \left|\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right|$$
$$= \left|\frac{v\,\Delta u - u\,\Delta v}{v\,(v+\Delta v)}\right| \leqslant \frac{|v|\,|\Delta u| + |u|\,|\Delta v|}{|v|\,|v+\Delta v|},$$

that is,

$$\left|\Delta\left(\frac{u}{v}\right)\right| \leqslant \frac{|v|\,|\Delta u| + |u|\,|\Delta v|}{|v|\,|v+\Delta v|}.$$

Dividing this inequality by $\left|\frac{u}{v}\right|$, we get the inequality

$$\left|\frac{\Delta\left(\frac{u}{v}\right)}{\left(\frac{u}{v}\right)}\right| < \left(\left|\frac{\Delta u}{u}\right| + \left|\frac{\Delta v}{v}\right|\right)\frac{|v|}{|v+\Delta v|}. \qquad (1)$$

Thus, the relative error of approximating the quotient of two numbers is estimated by Formula (1).

For sufficiently small Δv the ratio $\frac{|v|}{|v+\Delta v|}$ differs so little from 1 that in many cases it is regarded possible to write 1 in Formula (1) instead of $\frac{v}{|v+\Delta v|}$. In such cases, to estimate the relative error of approximating the quotient of two numbers, we use the inequality

$$\left|\frac{\Delta\left(\frac{u}{v}\right)}{\left(\frac{u}{v}\right)}\right| \leqslant \left|\frac{\Delta u}{u}\right| + \left|\frac{\Delta v}{v}\right|, \qquad (2)$$

which shows that the *relative error of approximating the quotient of the numbers u and v does not exceed the sum of relative errors of approximating these numbers.*

Henceforward, we shall use Inequality (2) to estimate the relative error of approximating the quotient of two numbers.

Note that when computing approximately the quotient of the numbers u and v, one should try to make the relative errors $\left|\frac{\Delta u}{u}\right|$ and $\left|\frac{\Delta v}{v}\right|$ equal.

Example. Calculate approximately the quotient $\frac{u}{v}$ of the numbers

$$u = 0.47145, \quad v = 15.732$$

with a relative error less than 0.02.

Solution. Find the approximations to the numbers u and v:

$$u = 4.7145 \cdot 10^{-1} \cong 4.71 \cdot 10^{-1} = 0.471,$$
$$v = 1.5732 \cdot 10 \cong 1.57 \cdot 10 = 15.7.$$

These approximate equalities are taken with relative errors not exceeding 0.01. Therefore the approximate equality

$$u \div v \cong (4.71 \cdot 10^{-1}) \div (1.57 \cdot 10) = 3 \cdot 10^{-2} = 0.03$$

takes place with a relative error less than 0.02. And the absolute error of this approximation is less than

$$\left|\frac{u}{v}\right| \cdot 0.02 < \frac{0.60}{15} \cdot 0.02 = 0.04 \cdot 0.02 = 0.0008.$$

SECTION TEST

QUESTIONS

1. How do we usually estimate the relative error of approximation of the quotient of two numbers?

EXERCISES

1. Compute approximately the quotient $\frac{a}{b}$ of the numbers $a = 0.12345678\ldots$ $b = 2.\overline{7}$ with a relative error less than 0.001.

2. Compute approximately the quotient $\frac{a}{b}$ of the numbers $a = 7.9\overline{17}$ and $b = 1.2345678910$ with a relative error less than 1/500.

ANSWERS

Chapter 1

Sec. 1.1. 1. 41 is a prime number, the rest of them being composite: $57 = 3 \cdot 19$; $1121 = 19 \cdot 59$; $793 = 13 \cdot 61$.

Sec. 1.2. 1. (a) 64; (b) 125; (c) 343; (d) 1,000,000. 2. (a) $256 = 2^8$; (b) $1024 = 2^{10}$.

Sec. 1.3. 1. (a) 1; 17; (b) 1; 3; 5; 9; 15; 45; (c) 1; 113; (d) 1; 2; 4; 7; 14; 17; 28; 34; 68; 119; 238; 476; (d) 1; 2; 4; 8; 16; 32. 2. (a) 19; (b) 2; 3; (c) 2; 7; (d) 2; 29.

Chapter 2

Sec. 2.1. 1. The fraction $\frac{17}{10}$ is irreducible, the fraction $\frac{396}{591}$ is irreducible. 2. (a) 73.98153; (b) 0.01291; (c) 0.198.

3. (a) $\frac{37}{1000}$; (b) $\frac{2503}{1000}$; (c) $\frac{713}{1000}$.

Sec. 2.2. 1. (a) 1.55; 0.14; 0.085; (b) 1.6; 1.5; 0.26; (c) 0.02; 0.6; 0.2. 2. $\frac{7}{20}$; $\frac{201}{100}$; $\frac{361}{50}$; $\frac{253}{250\,000}$. 3. Impossible: $\frac{1}{3}$; $\frac{12}{35}$; $\frac{5}{300}$; $\frac{1}{7}$; possible: $\frac{24}{15} : \frac{24}{15} = \frac{8}{5} = 1.6$.

Sec. 2.3. 1. (a) $0.\overline{8}$; (b) $0.1\overline{2}$; (c) $0.0\overline{17}$; (d) $6.\overline{0}$; (e) $0.\overline{45}$.

Sec. 2.4. 1. (a) $0.\overline{4}$; (b) 0.68; (c) $0.34\overline{12}$; (d) 0.3125.

2. (a) $0.1750\overline{}$; (b) $19.\overline{0}$; (c) $0.\overline{142857}$; (d) $0.\overline{27}$. 3. (a) $\frac{47}{2}$; (b) $\frac{1058}{45}$; (c) $\frac{56}{495}$; (d) $\frac{19}{1850}$.

Chapter 3

Sec. 3.1. 2. (a) $-0.\overline{428571}$; (b) $0.5625\overline{0}$; (c) $-5.\overline{67}$; (d) $0.\overline{0}$. 3. (a) $-\frac{157}{495}$; (b) $\frac{2}{1}$; (c) $-\frac{311}{90}$; (d) -0.

Sec. 3.2. 3. (a) $3.17\overline{2}$; (b) $0.00\overline{1}$; (c) $3.17\overline{2}$; (d) 0.

Sec. 3.3. 1. $f < l < c < b < a = d$. 2. $b > d > l > c > a$.

Sec. 3.4. 1. $a + b \cong 2.8$; 2.92; 2.93353; $a - b \cong 1.4$; 1.36; 1.35777. 2. $ab \cong 19.3$; $ab \cong 19.54$; $\frac{a}{b} \cong 0.228$; $\frac{a}{b} \cong 0.2310$.

Sec. 3.6. 1. (a) $c = 0.01$; (b) $c = 2.13$; (c) $c = 0$; (d) $c = -2$.

Chapter 4

Sec. 4.1. 1. (a) 1; (b) 343. 2. (a) a^{10}; (b) a^{12}. 3. (a) $a^{50} = (a^2)^{25}$; (b) $a^{50} = (a^5)^{10}$; (c) $a^{50} = (a^{10})^5$; (d) $a^{50} = (a^{25})^2$. 4. (a) $(ab)^8$; (b) 6^3; (c) $(xyz)^4$. 5. (a) $7^{10} = 7^2 \cdot 7^8$; (b) $a^6 = a^2 \cdot a^4$; (c) $(cd)^7 = c^7 \cdot d^7$. 6. (a) $5^6 = 5 \cdot 5^2 \cdot 5^3$; (b) $b^5 = b^2 \cdot b \cdot b^2$; (c) $(mn)^4 = (mn)^2 \cdot m^2 \cdot n^2$. 7. (a) $(abcd)^4 = a^4b^4c^4d^4$; (b) $(ab^2)^3 = a^3b^6$; (c) $6^2 = 2^2 \cdot 3^2$.

Sec. 4.2. 1. (a) 5; (b) $\frac{1}{10}$; (c) 1. 2. (a) 5^{-2}; (b) 7^3; (c) 1. 3. (a) $2.74 \cdot 10^1$; (b) $3.821 \cdot 10^3$; (c) $1.1 \cdot 10^{-3}$.

Sec. 4.3. 1. (a) $a^{50} = (a^{-2})^{-25}$; (b) $a^{50} = (a^{-5})^{-10}$; (c) $a^{50} = (a^{10})^5$; (d) $a^{50} = (a^{-25})^{-2}$. 2. (a) $(ab)^{-3}$; (b) $\left(\frac{7}{2}\right)^3$. 3. (a) $a^6 \cdot b^{-15}$; (b) $a^{14} \cdot b^{-4}$; (c) $a^{12} \cdot b^{20}$.

Chapter 5

Sec 5.1. 1. (a) Has no sense; (b) $-\frac{3}{7}$; (c) $\frac{54}{95}$.

Sec. 5.2. 1. abc. 2. (a) $5k$, where k is an integer; (b) $5k + 3$, where k is a nonnegative integer.

Sec. 5.4. 1. (a) a^6; (b) a^{10}. 2. (a) $\alpha^3\beta^2$; (b) $36a^3$.

Sec. 5.5. 1. (a) $2a^3b^2c$; (b) $-9a^2b^2$; (c) 0; (d) -60; (e) a^4.

Sec 5.6. (a) $abcd$; (b) $-\frac{1}{500}x^3y^2z$; (c) $-\frac{3}{25}x^5y^2z$; (d) $\alpha\beta$; (e) 0; (f) $-\frac{7}{13}$; (g) 0. 2. (a) $15abc$, the coefficient is 15, the degree is 3; (b) $abcd$; the coefficient is 1, the degree is 4; (c) $-5\alpha\beta\gamma$; the coefficient is -5, the degree is 3.

Sec. 5.7. 1. (a) Like monomials: a^2bc, $2abca = 2a^2bc$, $-3bca^2 = -3a^2bc$; (b) like monomials: $\alpha^3\beta$, $-\alpha\beta\alpha^2 = -\alpha^3\beta$, $7\alpha^2\beta\alpha = 7\alpha^3\beta$. 2. (a) 0; (b) $7\alpha^3\beta$; (c) 0. 3. (a) $-4abc$; (b) 0; (c) $-\alpha$.

Chapter 6

Sec. 6.1. 1. (a) $2x^2$; $-3xy$; $-x^3y$, $7y$; (b) $-x^7$; $-x^5$; $-2x^3$; $-3x$.

Sec. 6.2. 1. (a) $-x - x^2y$; (b) 0.

Sec. 6.3. 1. Only polynomial (a) has a standard form. 2. (a) $-a^3b + 3a^2 - 4b$; the degree of this polynomial is equal to 4; (b) $3x^5y + x^5 - 7y^2 + 2x - 1$; the degree of this polynomial is equal to 6; (c) $-\alpha\beta\gamma - 7\alpha^2 + 4\alpha\gamma$; the degree of this polynomial is equal to 3.

Sec. 6.4. 1. (a) $x^2 + 4x + 1$; (b) $a^5 - a^3 + 8a^2 + a$; (c) $x^2 - x + 5$; (d) 0. 2. (a) $-(-x^2 + y^2) + (2x - 1)$; (b) $-(-9y^2 + 1) + (-x^2 - 6y)$; (c) $-(a^3 + 3a^2) + (4 - a)$.

Sec. 6.5. 1. (a) $-a^2b^2c - a^2bc^2 - ab^2c^2$; (b) $a^2c + 12ac^2$; (c) $-2a^2b - 4ab^2 - 8b^3$. 2. (a) $a(a^2 + 4b^2)$; (b) $-2b(a^2 - 2ab + 2b^2)$; (c) $-5x^2y(xy + 1)$.

Sec. 6.6. 1. (a) $a^3 + 8b^3$; (b) $a^2 + 2ab + b^2 - c^2$; (c) $a^4 - 16b^4$.

Sec. 6.7. 1. (a) $x^2y + 2x$; (b) $a^2 + b^2 - c^2$; (c) $4\alpha\beta + 2\alpha\gamma$.

Sec. 6.8. 1. (a) -19; (b) $-4\frac{1}{4}$; (c) -5; (d) 3; (e) $\frac{11}{15}$; (f) 7.

Sec. 6.9. 1. (a) Yes; (b) yes. 2. (a) Yes; (b) yes.

Chapter 7

Sec. 7.1. 1. (a) $a^2 + 4ab + 4b^2$; (b) $4x^2 + 12xy + 9y^2$; (c) $10x^2 + 12xy + 10y^2$. 2. (a) $(a + 2c)^2$; (b) $(1 + x)^2$; (c) $(ac + d)^2$.

Sec. 7.2. 1. (a) $x^2 - 4xy + 4y^2$; (b) $\alpha^2\beta^2 - 2\alpha\beta\gamma + \gamma^2$; (c) $25x^2y^2 - 20xy + 4$. 2. (a) $(a - 2b)^2$; (b) $(x - 2)^2$; (c) $(a^2 - 1)^2$.

Sec. 7.3. 1. (a) $(x-2)^2+1$; (b) $\left(x+\frac{3}{2}\right)^2-\frac{13}{4}$; (c) $\left(x-\frac{5}{2}\right)^2-\frac{25}{4}$.

Sec. 7.4. 1. (a) $x^3 + 9x^2z + 27xz^2 + 27z^3$; (b) $a^3 + 6a^2 + 12a + 8$; (c) $8b^3 + 36b^2 + 54b + 27$. 2. (a) $(2x + y)^3$; (b) $(\alpha + 1)^3$; (c) $(3 + b)^3$.

Sec. 7.5. 1. (a) $x^3 - 3x^2 + 3x - 1$; (b) $8a^3 - 36a^2b + 54ab^2 - 27b^3$; (c) $a^6 - 6a^4b + 12a^2b^2 - 8b^3$. 2. (a) $(1 - x)^3$; (b) $(a - 2)^3$; (c) $(2a - 3b)^3$.

Sec. 7.6. 1. (a) $x^2 - 4$; (b) $a^4 - 1$; (c) $4\alpha^2 - 9\beta^2$. 2. (a) $(2a + 1)(2a - 1)$; (b) $(a + b)(a - b)$; (c) $(3x^2 + 2)(3x^2 - 2)$; (d) $(x^2 + 4)(x^2 - 4)$.

Sec. 7.7. 1. (a) $27x^3 + y^3$; (b) $a^{12} + 1$; (c) $8\alpha^3 + 27$. 2. (a) $(a^2 + b)(a^4 - a^2b + b^2)$; (b) $(3\alpha + 2\beta)(9\alpha^2 - 6\alpha\beta + 4\beta^2)$.

Sec. 7.8. 1. (a) $8a^3 - 27b^3$; (b) $a^6 - 1$; (c) $x^3y^3z^3 - t^3$. 2. (a) $(a - 2b)(a^2 + 2ab + b^2)$; (b) $(2a^3 - 3)(4a^4 + 6a^2 + 9)$; (c) $(xy - z^2)(x^2y^2 + xyz^2 + z^4)$.

Sec. 7.9. 1. (a) $a^8 - 17a^4 + 16$; (b) $a^2 + b^2 - c^2$; (c) $a^3 - b^3$. 2. (a) $(y - 5 + 2x)(y - 5 - 2x)$; (b) $3ab(a + b)$; (c) $(x - y)(x + y) \times (x^2 + y^2)(x^4 + y^4)(x^8 + y^8)$.

Sec. 7.10. 1. (a) $(a + c)(b + d)$; (b) $(a - b + c)(a - b - c)$; (c) $(a - 2b)(a + 2b)(a^2 + 4b^2)$; (d) $(a + b)(a + b + c)$; (c) $(3y - 1 + x)(3y - 1 - x)$; (f) $(x^2 + y - 1)(x^2 - y + 5)$.

Chapter 8

Sec. 8.1. 1. (a) $a - b$; (b) $a^2 - ab + b^2$; (c) $\alpha - \beta$.

Sec. 8.2. 1. (a) $2 \cdot \frac{a^2+b^2}{a^2-b^2}$; $4 \cdot \frac{ab}{a^2-b^2}$;

(b) $\frac{3\alpha^3-\alpha^2\beta+2\beta^3}{2\alpha^2}$; $\frac{-\alpha^3-\alpha^2\beta-2\beta^3}{2\alpha^2}$.

2. (a) $\frac{(a-b)b}{a}$; (b) $\frac{1}{a^4+a^2b^2+b^4}$.

Sec. 8.3. 1. (a) $\frac{a(a-b)}{b(a-b)}$ and $\frac{b^2}{b(a-b)}$; (b) $\frac{(\alpha+\beta)^2}{\alpha^2-\beta^2}$ and $\frac{\alpha^3}{\alpha^2-\beta^2}$.

Sec. 8.4. 1. (a) b; (b) α. 2. Expressions (b), (c), (d) have no sense.

Sec. 8.5. 1. (a) $a \neq b$ and $a \neq 0$; (b) $x \neq 0$, $y \neq 0$ and $x \neq -y$; (c) $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq \beta$.

2. (a) $\frac{400}{111}$; (b) $28\frac{1}{111}$; (c) $-\frac{5}{3}$.

Sec. 8.6. 1. (a) $a \neq b$ and $a \neq -b$; (b) $a \neq b$ and $a \neq -b$.

Chapter 9

Sec. 9.1. 2. (a) $\frac{7}{3}$; (b) 0; (c) $-\frac{5}{4}$; (d) $-\frac{15}{4}$.

Sec. 9.2. 2. (a), (b), (c) yes; (d), (e) no.

Sec. 9.3. 1. (a) 7; (b) $-\frac{6}{5}$; (c) no roots; (d) -6; (c) any real number is a root of this equation; (f) 0; (g) $\frac{5}{3}$; (h) $\frac{4}{5}$; (i) 2.

Sec. 9.4. 1. 10. 2. 20 km/h.

Chapter 11

Sec. 11.1. 1. (a) $\frac{1}{13}$ and $-\frac{1}{13}$; (b) 25 and -25; (c) 1000 and -1000; (d) 1.2 and -1.2; (e) 6.5 and -6.5; (f) 0.16 and -0.16; (g) 0; (h) no.

Sec. 11.2. 1. (a) $\sqrt{3}$; (b) 0.7; (c) $\frac{1}{3}$; (d) 0; (e) no.

2. (a) $\frac{1}{13}$; (b) 6.5; (c) 0.11.

Sec. 11.4. 1. (a) 1.732; (b) 2.236; (c) 2.645. 2. 1.41421.

Sec. 11.5. 1. (a) $11\sqrt{3}$; (b) $7\sqrt{3}$; (c) $3\sqrt{26}$; (d) $10\sqrt{3} - \frac{\sqrt{2}}{2}$. 2. (a) -14; (b) $\sqrt{2}$.

Chapter 12

Sec. 12.1. 1. (a), (b), and (c). 2. (a) No; (b) yes. 3. (a); (b) yes; (c) no.

Sec 12.2. 1. (a) $x_1=x_2=0$; (b) no roots; (c) $x_1=\sqrt{\frac{1}{7}}$, $x_2=-\sqrt{\frac{1}{7}}$; (d) no roots; (e) $x_1=\sqrt{\frac{5}{3}}$; $x_2=-\sqrt{\frac{5}{3}}$; (f) $x_1=0$; $x_2=\frac{4}{3}$; (g) $x_1=0$, $x_2=-2$.

Sec. 12.3. 1. (a) $x_1=1$, $x_2=\frac{4}{3}$; (b) $x_1=1$, $x_2=6$; (c) $x_1=2$; $x_2=\frac{3}{2}$; (d) $x_1=x_2=\frac{3}{5}$.

Sec. 12.4. 1. (a) $x_1=9$, $x_2=-1$; (b) $x_1=x_2=3$; (c) no roots.
Sec. 12.5. 1. 11 and 12. 2. 32.

Chapter 13

Sec. 13.1. 1. (a) $x_1=\sqrt{3}$, $x_2=-\sqrt{3}$, $x_3=\sqrt{2}$, $x_4=-\sqrt{2}$; (b) no roots; (c) $x_1=-\sqrt{2}$, $x_2=\sqrt{2}$; (d) $x_1=\frac{2}{\sqrt{3}}$; $x_2=-\frac{2}{\sqrt{3}}$; (e), (f) no roots.

Sec. 13.2. 1. (a) $x_1=\sqrt{3}$, $x_2=-\sqrt{3}$, $x_3=-1$; (b) $x_1=1$, $x_2=-1$ $x_3=3$, $x_4=-3$; (c) $x_1=0$.
Sec. 13.3. 1. (a) $x=-3$; (b) $x=-1$; (c) no roots.

Sec. 13.4. 1. (a) $x_1=2$, $x_2=-4$; (b) $x_1=\sqrt{5}$, $x_2=-\sqrt{5}$; (c) $x=-\frac{5}{2}$.

Sec. 13.5. 1. 14 km/h. 2. 150 metres per month.

Chapter 14

Sec. 14.1. 3. (0, 1); (−1, 0); (1, 2); (2, 3); (3, 5). 4. Infinitely many.

Sec. 14.2. 2. (a) No; (b) yes. 3. (a) $\left(-\frac{1}{2}, -\frac{5}{3}\right)$; (b) $\left(-2, \frac{3}{2}\right)$; (c) (20, 7); (d) no solutions.

Sec. 14.3. 1. (a) $\left(-\frac{25}{17}, \frac{23}{17}\right)$; (b) no solutions; (c) infinitely many solutions.

Sec. 14.4. 1. (a) $\left(\frac{2}{9}, \frac{23}{9}\right)$; (b) $\left(\frac{2}{3}, 0\right)$; (c) infinitely many solutions.
Sec. 14.5 1. (a). Equivalent; (b) not equivalent.
Sec. 14.6. 1. 9 and 6. 2. 40 and 25 m.

Chapter 15

Sec. 15.1. 1. (a) $y=\frac{1}{2}x$; (b) $y=-3x$; (c) $y=-\frac{1}{2}x$; (d) $y=5x$.

Sec. 15.2. 2. (a) $\left(-\frac{2}{3}; 0,\right)$; (b) $\left(\frac{2}{3}, 0\right)$; (c) (−3, 0); (d) (2, 0); (e) (8, 0); $\left(\frac{5}{3}, 0\right)$.

3. Parallel lines: (a) and (b). 4. (a) $y = 2x + 3$; (b) $y = 2x + 3$; (c) $y = \frac{3}{2}x + 3$; (d) $y = -x + 3$.

Sec. 15.3. 1. (a) $s = 4t + 5$; (b) $s = 6t + 2$.

Sec. 15.4. 1. (a) (0, 1); (b) no solutions; (c) solutions are represented by all pairs of coordinates (x, y) of the points belonging to the straight line $y = -3x + 1$.

Chapter 16

Sec. 16.1. 1. (a) $\left(-\frac{11}{13}, -\frac{6}{13}, \frac{17}{13}\right)$; (b) (7, 5, 5).

Sec. 16.2. 1. The distance from A to B is equal to 7.2 km; the rate of flow of the river is 0.8 km/h; the speed of the boat in stagnant water is equal to 4 km/h.

Sec. 16.3. 1. (a) (7, 10), (−3, 0); (b) (3, −3, 1), (3, 1, −3).

Sec. 16.4. 1. 3 and 5. 2. 5 and 20.

Sec. 16.5. 1. The speed of the motorcyclist is 50 km/h; the speed of the car is 75 km/h; the car left the point B 30 minutes later than the motorcyclist started from the point A. 2. The length of the path is 8 km, the speed is 1 km/h. 3. The distance is equal to 290 km; the rate of flow of the tributary is 2 km/h.

Chapter 17

Sec. 17.1. 1. (a) $x^2 + 8x + 15 = (x + 3)(x + 5)$; (b) $4x^2 - 4x + 1 = 4\left(x - \frac{1}{2}\right)^2$; (c) not factorable since $D < 0$.

Sec. 17.2. 1. (a) $x^2 + x$; (b) $x^2 - \frac{13}{4}x - 3$; (c) $x^2 + 2x - 6$. 2. (a) Both roots are positive; (b) the roots have unlike signs.

Sec. 17.5. 1. (a) $y = (x - 1)^2 - 1$; (b) $y = -(x - 1)^2 - 1$; (c) $y = 2(x - 1)^2 - 1$; (d) $y = -2(x - 1)^2 - 1$.

Chapter 18

Sec. 18.1. 1. (a) $\Delta s = 4t\Delta t + 2\Delta t^2$, $v_{av} = 4t + 2\Delta t$, $v = 4t$, $v(1) = 4$; (b) $\Delta s = 3t^2\Delta t + 3t\Delta t^2 + \Delta t^3$, $v_{av} = 3t^2 + 3t\Delta t + \Delta t^2$, $v = 3t^2$; $v(1) = 3$; (c) $\Delta s = 6\Delta t$, $v_{av} = 6$, $v = 6$, $v(1) = 6$; (d) $\Delta s = 6t\Delta t + 3\Delta t^2$, $v_{av} = 6t + 3\Delta t$, $v = 6t$, $v(1) = 6$.

Sec. 18.2. 1. (a) 3; (b) $6x + 2$; (c) $21x^2 - 4x + 3$; (d) $4x^3$; (e) $5x^4 + 4x^3 - 3x^2$. 2. (a) $v(t) = 3$, $v(2) = 3$; (b) $v(t) = 6t$, $v(2) = 12$; (c) $v(t) = 15t^2 - 4t$, $v(2) = 52$.

Sec. 18.3. 1. (a) $2x + C$; (b) $\frac{3}{2}x^2 - 2x + C$; (c) $\frac{x^2}{2} + x + C$; (d) $\frac{2}{3}x^3 + 4x + C$; (e) $\frac{x^3}{3} - \frac{x^2}{2} + 2x + C$.

Sec. 18.4. 1. (a) $s = 5t^2 - 10t$; (b) 20 m; (c) $v = 0$; (d) $t = 1$; (c) $t = 2$; (f) 20 m/s.

Chapter 19

Sec. 19.1 1. (a) $(-\infty, 5)$; (b) $\left(-\infty, -\frac{4}{3}\right)$; (c) $\left(-\infty, -\frac{7}{2}\right)$.

Sec. 19.2. 1. (a) $\left(-\infty, -\frac{5}{2}\right)$; (b) $(-\infty, -4)$; (c) $\left(\frac{2}{3}, -\infty\right)$.

19.3. 1. (a) $\left(\frac{1}{2}; +\infty\right)$; (b) $(-\infty, -1)$; (c) $(-\infty, +\infty)$.

Sec. 19.4. 1. (a) $\left(-\frac{5}{3}, 3\right)$; (b) no solutions; (c) $(-\infty, -7)$.

Chapter 20

Sec. 20.1. 1. (a) Yes; (b) no; (c) yes.

Sec. 20.2. 1. (a) $\left(-\frac{1}{3}, 2\right)$; (b) $(-\infty, -1)$ and $(2, +\infty)$; (c) $(-\infty, -2)$ and $(3, +\infty)$.

Sec. 20.3. 1. (a) $\left(-\infty, \frac{4}{5}\right)$ and $\left(\frac{4}{5}, +\infty\right)$; (b) no solutions; (c) no solutions.

Sec. 20.4. 1. (a) No solutions; (b) $(-\infty, +\infty)$; (c) $(-\infty, +\infty)$.

Sec. 20.5. 1. (a) $(-\infty, 2)$ and $(2, +\infty)$; (b) $(-\infty, +\infty)$; (c) $\left(-\infty, -\frac{1}{3}\right)$ and $(2, +\infty)$.

Chapter 21

Sec. 21.1. 1. (a) (1, 2) and $(3, +\infty)$; (b) $(-3, -1)$ and (1, 2); (c) $(-2, -1$ and (1, 4).

Sec. 21.2. 1. (a) $(-\infty, -1)$ and (2, 5); (b) $(-\infty, -1)$ and (2, 4); (c) $(-2, 1)$ and (2, 3).

Sec. 21.3. 1. (a) $(-5, -3)$, $(-2, -1)$ and (0, 1); (b) no solutions; (c) $(-\infty, -2)$, $(-1, 1)$ and $(2, +\infty)$.

Sec. 21.4. 1. (a) $\left[-\frac{5}{2}, +\infty\right)$; (b) $(-\infty, -3]$ and $[7, +\infty)$; (c) $[-4, -1]$ and $(5, +\infty)$.

Chapter 23

Sec. 23.1 3. 10.

Sec. 23.2. 1. (a) $2\sqrt[3]{4}$; (b) $2\sqrt[5]{25}$; (c) $97\frac{1}{\sqrt[3]{12}}$; (d) $2\sqrt[3]{6}-\sqrt[3]{75}+2\sqrt[3]{4}-2\sqrt[3]{5}$.

Sec. 23.3. 1. (a) $2\sqrt[4]{5}$; (b) $3\sqrt[4]{5}$; (c) $12-3\sqrt{17}$; (d) -0.1.

Sec. 23.4. 1. (a) $\sqrt[8]{2^7}$; (b) $\sqrt[6]{2}$; (c) $4\sqrt[12]{2^7}$. 2. (a) $\sqrt[6]{9}$, $\sqrt[6]{8}$, and $\sqrt[6]{5}$; (b) $\sqrt[8]{625}$, $\sqrt[8]{225}$, and $\sqrt[8]{50}$.

Sec. 23.5. 1. (a) 1.442; (b) 1.709; (c) 1.912, 2. (a) 1.1; (b) 1.4.

Chapter 25

Sec. 25.1. 1. $a_5 = -45.2$, 2. $S_{30} = 1230$. 3. Yes, $34 = a_{28}$, $a_1 = -47$, $d = 3$.

Sec. 25.2. 1. $a_{15} = -10^{11}$. 2. $S_{16} = \frac{2^{16}-1}{50}$.

Sec. 25.3. 1. The speed of the cyclist is 30 km/h, of the motorcyclist 40 km/h; of the car 50 km/h. 2. $2^{64} - 1$.

Chapter 26

Sec. 26.1. **1.** (a) 10; (b) 100; (c) 1000; (d) 10,000; (e) 100,000. **2.** (a) 5; (b) 32; (c) 48; (d) 7.

Sec. 26.2. **1.** (a) 10 111; (b) 111 110; (c) 11 101 000; (d) 110 100 101.

Sec. 26.3. **1.** (a) 11 100; (b) 11; (c) 100; (d) 101. **2.** (a) 10; (b) 10; (c) 1110. **3.** (a) 1111; (b) 100 011; (c) 11 110.

Chapter 27

Sec. 27.2. **1.** With deficit: (a) 37.578; (b) 0.002; (c) −117.010; (d) 0.399; (e) −31.722; (f) 0.000; (g) 32.100; (h) −2.379; with excess: (a) 37.579; (b) 0.003; (c) −117.009; (d) 0.400; (e) −37.721; (f) 0.001; (g) 32.101; (h) −2.378. **2.** (a) 37.579; (b) 0.003; (c) −117.010; (d) 0.400; (e) −37.721; (f) 0.000; (g) 32.100; (h) −2.379.

Sec. 27.3. **1.** (a) $\alpha + \beta \cong 19.346$ and $\alpha - \beta \cong 5.370$; (b) $\alpha - \beta \cong 7.991$ and $\alpha - \beta \cong 6.353$; (c) $\alpha + \beta \cong 14.430$ and $\alpha - \beta \approx 7.856$; (d) $\alpha + \beta \cong 4.352$ and $\alpha - \beta \cong 1.894$; (e) $\alpha + \beta \cong -3.902$ and $\alpha - \beta \cong 38.370$. Approximation error is 0.001. **2.** Three, four, five. **3.** (a) $\alpha + \beta \cong 5.25$; $\alpha - \beta \cong 1.07$; (b) $\alpha + \beta \cong 14.34$; $\alpha - \beta \cong 20.31$; (c) $\alpha + \beta \cong -9.98$, $\alpha - \beta \cong 4.02$.

Sec. 27.4. **1.** 8.5.2. $0.38 + 3.79 + 7.18 + 11.53 + 2.24 + 4.25 + 9.10 = 38.47$. **3.** The error does not exceed 0.0035.

Sec. 27.5. **1.** (a) 0.1; (b) 0.1; (c) 1; (d) 1. **2.** (a) 71,523 ≅ 71,500; the absolute error is not greater than 100; the relative error is not greater than 0.01; (b) 0.568 ≅ 0.5; the absolute error is not greater than 0.1; the relative error is not greater than 1; (c) 0.00328 ≅ 0.003; the absolute error is not greater than 0.001; the relative error is not greater than 1.

Sec. 27.6. **1.** $a \cong 0.1235$; $b \cong 2.778$; $ab \cong 0.3430830$. **2.** $a \cong 7.917$, $b \cong 1.235$, $ab \cong 9.777$.

Sec. 27.7. **1.** $a \cong 0.1235$; $b \cong 2.778$; $\frac{a}{b} \cong 0.0445$. **2.** $a \cong 7.917$. $b \cong 1.235$, $\frac{a}{b} \cong 6.410$.

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To the Reader

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MATHEMATICS FOR TECHNICAL SCHOOLS. WORKED EXAMPLES AND PROBLEMS WITH ELEMENTS OF THEORY

N. Bogomolov

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